Buckingham's pi-theorem

Harald Hanche-Olsen
hanche@math.ntnu.no

Theory

This note is about physical quantities $R_1, \dots, R_n$. We like to measure them in a *consistent* system of units, such as the SI system, in which the basic units are the meter, kilogram, second, ampere, and kelvin (m, kg, s, A, K).[1] As it will turn out, the existence of consistent systems of measurement has nontrivial consequences.

We shall assume the fundamental units of our system of units are $F_1, \dots, F_m$, so that we can write

$$R_j = v(R_j)[R_j] = \rho_j [R_j] \tag{1}$$

where $\rho_j = v(R_j)$ is a number, and $[R_j]$ the units of R_j. We can write $[R_j]$ in terms of the fundamental units as a product of powers:

$$[R_j] = \prod_{i=1}^{m} F_i^{a_{ij}} \qquad (j = 1, \dots, n),.$$

It is also important for the fundamental units to be *independent* in the sense that

$$\prod_{i=1}^{m} F_i^{x_i} = 1 \Rightarrow x_1 = \cdots = x_m = 0. \tag{2}$$

We shall not be satisfied with just one system of units: The whole crux of the matter hinges on the fact that our choice of fundamental units is quite arbitrary. So we might prefer a different system of units, in which the units F_i are replaced by $\hat{F}_i = x_i^{-1} F_i$. Here x_i can be an arbitrary positive number for $i = 1, \dots, m$. We can also write our quantities in the new system thus: $R_j = \hat{v}(R_j)[R_j]\hat{} = \hat{\rho}_j [R_j]\hat{}$. We compute

$$R_j = v(R_j) F_1^{a_{1j}} \cdots F_m^{a_{mj}} = \underbrace{v(R_j) x_1^{a_{1j}} \cdots x_m^{a_{mj}}}_{\hat{v}(R_j)} \hat{F}_1^{a_{1j}} \cdots \hat{F}_m^{a_{mj}}$$

[1] Perhaps we should also include the mole (mol) and candela (cd).

from which we deduce the relation

$$\hat{\rho}_j = \rho_j \prod_{i=1}^{m} x_i^{a_{ij}}. \tag{3}$$

For example, if $F_1 = \mathrm{m}$ and $F_s = \mathrm{s}$, and R_1 is a velocity, then $[R_1] = \mathrm{m\,s^{-1}} = F_1 F_2^{-1}$ and so $a_{11} = 1$, $a_{21} = -1$. With $\hat{F}_1 = \mathrm{km}$ and $\hat{F}_2 = \mathrm{h}$, we find $x_1 = 1/1000$ and $x_2 = 1/3600$, and so $\hat{\rho}_1 = \rho_1 \cdot 3.6$. Hence the example $\rho_1 = 10$, $\hat{\rho}_1 = 36$ corresponds to the relation $10\,\mathrm{m/s} = 36\,\mathrm{km/h}$.

We define the *dimension matrix* A of $R_1, \ldots, R_n$ by

$$A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix}.$$

It is now time to introduce the heroes of our drama: The *dimensionless combinations* of the variables R_j. A *combination* of these variables is merely a product of powers: $R_1^{\lambda_1} \cdots R_n^{\lambda_n}$. We compute the units of this combination:

$$[R_1^{\lambda_1} \cdots R_n^{\lambda_n}] = \prod_{i=1}^{m} F_i^{a_{i1}\lambda_1 + \cdots + a_{in}\lambda_n}. \tag{4}$$

We call the combination *dimensionless* if this unit is 1; thus we arrive at the important result that this is equivalent to $A\lambda = 0$, where we write $\lambda = (\lambda_1, \ldots, \lambda_n)^{\mathrm{T}}$. *There is, therefore, a 1–1 correspondence between the null space* $\mathcal{N}(A)$ *and the set of dimensionless combinations of the variables.*

It may not come as a big surprise that dimensionless combinations have a value independent of the system of units. We simply use (3) and compute:

$$\begin{aligned} \prod_{j=1}^{n} \hat{\rho}_j^{\lambda_j} &= \prod_{j=1}^{n} \Bigl(\rho_j \prod_{i=1}^{m} x_i^{a_{ij}}\Bigr)^{\lambda_j} = \Bigl(\prod_{j=1}^{n} \rho_j^{\lambda_j}\Bigr) \prod_{j=1}^{n} \prod_{i=1}^{m} x_i^{a_{ij}\lambda_j} \\ &= \Bigl(\prod_{j=1}^{n} \rho_j^{\lambda_j}\Bigr) \prod_{i=1}^{m} \prod_{j=1}^{n} x_i^{a_{ij}\lambda_j} = \prod_{j=1}^{n} \rho_j^{\lambda_j} \end{aligned}$$

since $A\lambda = 0$ implies $\prod_{j=1}^{n} x_i^{a_{ij}\lambda_j} = 1$.

Moreover, if you pick a basis for $\mathcal{N}(A)$ and take the corresponding dimensionless combinations, $\pi_1, \ldots, \pi_{n-r}$ (here r is the rank of A), then any dimensionless combination can be written as a product $\pi_1^{c_1} \cdots \pi_{n-r}^{c_{n-r}}$, where the exponents are uniquely given (they are the coefficients of a member of $\mathcal{N}(A)$ in the chosen basis). We shall call this a *maximal set of independent dimensionless combinations.* We can now state

1 Theorem. (Buckingham's pi-theorem)
Any physically meaningful relation $\Phi(R_1,\dots,R_n)=0$*, with* $R_j \neq 0$*, is equivalent to a relation of the form* $\Psi(\pi_1,\dots,\pi_{n-r})=0$ *involving a maximal set of independent dimensionless combinations.*

The important fact to notice is that the new relation involves r fewer variables than the original relation; this simplifies the theoretical analysis and experimental design alike.

We are not quite ready to prove this, however. Amongst other things, we must give a precise meaning to the phrase "physically meaningful."

First of all, Φ must also have units, and a value:

$$[\Phi]=\prod_{i=1}^{m} F_i^{b_i}. \tag{5}$$

The value is given by just inserting the values of R_j in the formula for Φ and computing:

$$v\big(\Phi(R_1,\dots,R_n)\big)=\Phi\big(v(R_1),\dots,v(R_n)\big)$$

Furthermore, when we change to a different set of units, the value of Φ must change according to a law similar to (3). Thus we get

$$\begin{aligned}\Phi\big(\hat{v}(R_1),\dots,\hat{v}(R_n)\big)&\\ &=\hat{v}\big(\Phi(R_1,\dots,R_n)\big)\\ &=x_1^{b_1}\cdots x_m^{b_m}\,v\big(\Phi(R_1,\dots,R_n)\big)\\ &=x_1^{b_1}\cdots x_m^{b_m}\,\Phi\big(v(R_1),\dots,v(R_n)\big)\end{aligned}$$

and therefore

$$\Phi(x_1^{a_{11}}\cdots x_m^{a_{m1}}\rho_1,\dots,x_1^{a_{1n}}\cdots x_m^{a_{mn}}\rho_n)=x_1^{b_1}\cdots x_m^{b_m}\Phi(\rho_1,\dots,\rho_n) \tag{6}$$

for all real $\rho_1,\dots,\rho_n$ and positive $x_1,\dots,x_m$. It is this relation we shall think of when we say *physically meaningful* in Buckingham's pi-theorem. We shall, however, have to insist on one more feature: Since Φ is supposed to combine the quantities R_j, the units of Φ must be the units of some combination of the variables R_j.

We now begin the proof. First note that, by the final statement of the above paragraph, we may as well replace Φ by $R_1^{c_1}\cdots R_n^{c_n}\Phi(R_1,\dots,c_1)$ where the coefficients $c_1,\dots,c_n$ are chosen so that the new function is dimensionless – that is, $b_1=\cdots=b_m=0$ in (5).

The dimension matrix A, having the rank r, has r linearly independent columns. We may as well assume these are the first r columns, corresponding to the variables $R_1, \dots, R_r$.[2] Then $R_1, \dots, R_r$ are *dimensionally independent* in the sense that their only dimensionless combination is the trivial one: $R_1^{\lambda_1} \cdots R_r^{\lambda_r}$ is dimensionless only if $\lambda_1 = \cdots = \lambda_r = 0$ (this follows immediately from (4)).

I claim a natural 1–1 correspondence:

$$(R_1, \dots, R_n) \longleftrightarrow (R_1, \dots, R_r, \pi_1, \dots, \pi_{n-r})$$

Clearly, the only possible difficulty here is expressing R_k (where $k > r$) in terms of the quantities on the right-hand side. But linear algebra tells us that column k of A is a linear combination of the first r columns, and so $[R_k] = [R_1^{c_1} \cdots R_r^{c_r}]$ for a suitable choice of $c_1, \dots, c_r$. But then $R_k R_1^{-c_1} \cdots R_r^{-c_r}$ is dimensionless, so it can be written $\pi_1^{d_1} \cdots \pi_{n-r}^{d_{n-r}}$. Therefore we can write $R_k = R_1^{c_1} \cdots R_r^{c_r} \pi_1^{d_1} \cdots \pi_{n-r}^{d_{n-r}}$.

Now, using the above 1–1 correspondence, write

$$\Phi(R_1, \dots, R_n) = \psi(R_1, \dots, R_r, \pi_1, \dots, \pi_{n-r}) \tag{7}$$

for a suitable function ψ.

In a moment, I shall prove that $\psi(R_1, \dots, R_r, \pi_1, \dots, \pi_{n-r})$ does in fact not depend on $R_1, \dots, R_r$. Thus we may write

$$\psi(R_1, \dots, R_r, \pi_1, \dots, \pi_{n-r}) = \Psi(\pi_1, \dots, \pi_{n-r})$$

and the proof of Buckingham's pi-theorem will be complete.

To prove the independence of $R_1, \dots, R_r$, replace each R_j in (7) by its value ρ_j and substitute this in both sides of (6), and remember that $b_i = 0$:

$$\begin{aligned} &\psi(x_1^{a_{11}} \cdots x_m^{a_{m1}} \rho_1, \dots, x_1^{a_{1r}} \cdots x_m^{a_{mr}} \rho_r, \pi_1, \dots, \pi_{n-r}) \\ &\qquad = \psi(\rho_1, \dots, \rho_r, \pi_1, \dots, \pi_{n-r}). \end{aligned}$$

Now, I claim that, given positive numbers $\rho_1, \dots, \rho_r$, we can pick positive numbers $x_i, \dots, x_m$ so that the numbers $x_1^{a_{1j}} \cdots x_m^{a_{mj}} \rho_j$ (for $j = 1, \dots, r$) on the left-hand side of the above equation can be any given positive numbers. To be specific, we can make them all equal to 1. That is, we can solve the equations

$$\prod_{i=1}^{m} x_i^{a_{ij}} = 1/\rho_j, \quad j = 1, \dots, r$$

[2] If not, we just renumber the variables to make it thus.

with respect to x_i. In fact if we write $x_i = \exp(\xi_i)$ the above equation is equivalent to

$$\sum_{i=1}^{m} a_{ij}\xi_i = -\ln\rho_j, \quad j = 1,\ldots,r.$$

This equation is solvable because the left $m \times r$ submatrix of A has rank r, and therefore its rows span $\mathbb{R}^r$. This proves the claim above, and therefore the theorem. ∎

Practice

Pipe flow. We consider the problem of determining the pressure drop of a fluid flowing through a pipe. If the pipe is long compared to its diameter, we shall assume that the pressure drop is proportional to the length of the pipe, all other factors being equal. Thus we really look for the (average) pressure gradient ∇P, and presume the length of the pipe to be irrelevant.

Variables that *are* relevant clearly include other properties of the pipe: Its diameter D, and its roughness e. To a first approximation, we just let e be the average size of the unevennesses of the inside surface of the pipe; thus it is a length.

Also relevant are fluid properties. We shall use the kinematic viscosity $\nu = \mu/\rho$ together with the density ρ. In a Newtonian fluid in shear motion, the shear tension (a force per unit area) is proportional to a velocity gradient, and the dynamic viscosity μ is the required constant of proportionality: Thus the units of μ are $\mathrm{N\,m^{-2}/s^{-1}} = \mathrm{kg\,m^{-1}s^{-1}}$, and therefore the units of ν are $\mathrm{m^2s^{-1}}$.

Finally, the average fluid velocity v is most definitely needed.

The dimension matrix can be written as follows.

	∇P	v	D	e	ν	ρ
m	−2	1	1	1	2	−3
kg	1	0	0	0	0	1
s	−2	−1	0	0	−1	0

We can find the null space of this, and hence use it to find the dimensionless combinations. However, it is in fact easier to find dimensionless combinations by inspection. It is easy to see that the matrix has rank 3, so with 6 variables, we must find $6 - 3 = 3$ independent dimensionless combinations. There are, of course, an infinite number of possibilities, since the choice corresponds to choosing a basis for the nullspace of A. In this case we may be guided by common practice, however, and pick dimensionless quantities as follows:

Reynold's number	$\mathrm{Re} = \dfrac{vD}{\nu}$
Relative roughness	$\varepsilon = \dfrac{e}{D}$
(no name)	$\dfrac{\nabla P \cdot D}{\rho v^2}$

Since we expect the ∇P to be a function of the other variables, we should have a relationship between the above quantities which has a unique solution for the only variable containing ∇P:

$$\frac{\nabla P \cdot D}{\rho v^2} = f(\mathrm{Re}, \varepsilon),$$

which we write as

$$\nabla P = \frac{2\rho v^2}{D} f(\mathrm{Re}, \varepsilon).$$

The extra factor 2 is there because then f is known as *Fanning's friction factor*. Presumably, Fanning used the radius of the pipe rather than the diameter as the basis for his analysis.

(A copy and description of the Moody diagram should be included here.)

One final remark. We did not really need to write down the dimension matrix. It is quite clear that the three dimensionless quantities we found are independent, since each of them contains at least one variable which is not present in the two others. Since there were only three fundamental units involved, the dimension matrix could not possibly have rank greater than 3, and therefore there could not exist more than three independent dimensionless combinations. Still, the dimension matrix provides a convenient way to summarize the dimensions and to reduce everything to a problem in linear algebra.

Water waves. We consider surface waves in water. These waves can be conveniently characterized by a wave number $k = 2\pi/\lambda$ (where λ is the wavelength) and an angular frequency ω. We seek a *dispersion relation* expressing ω as a function of ω. Presumably, the depth d plays a role, as well as the wave height h, the acceleration of gravity g, and the fluid properties: the density ρ and (for very small waves) the surface tension τ. We shall assume that the viscosity is negligible.

The dimensions of all these variables can be summarized as follows.

Variable	ω	k	h	d	ρ	τ	g
Units	s^{-1}	m^{-1}	m	m	$\mathrm{kg\,m}^{-3}$	$\mathrm{N\,m}^{-1} = \mathrm{kg\,s}^{-2}$	$\mathrm{m\,s}^{-2}$

With no less than seven variables, and three fundamental units, we expect to find four independent dimensionless combinations. One reasonable choice is (Bo is the *Bond number*):

$$hk, \quad dk, \quad \frac{\omega^2}{gk}, \quad \mathrm{Bo} = \frac{\rho g}{\tau k^2}.$$

A relationship between all these, solved for the one combination that involves ω, then leads to a relationship of the form

$$\omega^2 = gk\Psi\Big(hk, dk, \mathrm{Bo}\Big).$$

We see that, for example, when waves are long, Bo is large, so we may ignore the influence of surface tension. If the water is deep compared to the wave length then $dk \approx \infty$, while if the wave height is small compared to wave length, $hk \approx 0$. When all of these approximations hold, then, we expect Ψ to be roughly constant, so ω^2 is proportional to gk. In fact we find $\omega^2 = gk$ in the limit, but this requires more detailed analysis.

For very short waves (ripples) in deep water, it seems reasonable to assume that only surface tension is responsible for the wave motion, so that g does not enter the problem. You could do a new dimensional analysis under this assumption, but it is easier to see directly that Ψ must be a linear function of its last argument for g to cancel out. If we still assume $dk \gg 1$ and $hk \ll 1$, we end up with a relationship of the form

$$\omega^2 = \frac{\tau k^3}{\rho}$$

except the right-hand side should be multiplied by a dimensionless constant. But again, this constant turns out to be 1.