Solution set 2

to some problems given for TMA4230 Functional analysis

2005–02–20

Problem 4.5.5. The short version: $\big((ST)^*f\big)(x) = f\big(STx\big) = \big(S^*f\big)(Tx) = \big(T^*(S^*f)\big)(x) = \big((T^*S^*)f\big)(x)$. But perhaps it is more instructive to note that the definition of the adjoint can be written $S^*f = f \circ S$, where $\circ$ denotes the composition of functions. (When we write ST, that is really short for $S \circ T$.) So the identity we are asked to show is nothing but the obvious $f \circ (T \circ S) = (f \circ T) \circ S$.

Problem 4.5.8. In our notation, we are asked to prove $(T^*)^{-1} = (T^{-1})^*$. More precisely, assuming $T \in B(X,Y)$ is invertible, T^* is also invertible, with inverse $(T^{-1})^*$.

But equation (11) says $(ST)^* = S^*T^*$. Apply with $S = T^{-1}$ to get $(T^{-1})^*T^* = I^* = I$. And apply that with T and T^{-1} interchanged, to get $T^*(T^{-1})^* = I$. Together, these two show that T^* and $(T^{-1})^*$ are each other's inverses.

Problem 4.5.9. Note that any set and its closure have the same annihilator: Just recall that the null space of any bounded linear functional is closed. So what we are asked to prove is just[1]

$$\mathcal{R}(T)^\perp = \mathcal{N}(T^*).$$

Now, if $f \in Y^*$ then[2]

$$\begin{aligned} f \in \mathcal{R}(T)^\perp &\Leftrightarrow f(Tx) = 0 \qquad \forall x \in X \\ &\Leftrightarrow (T^*f)(x) = 0 \qquad \forall x \in X \\ &\Leftrightarrow T^*f = 0 \\ &\Leftrightarrow f \in \mathcal{N}(T). \end{aligned}$$

Problem 4.5.10. Take a typical element Tx of $\mathcal{R}(T)$, where $x \in X$. We need to show that[3] $Tx \in \mathcal{N}(T^*)_\perp$. Thus, we take $f \in \mathcal{N}(T^*)$, and must prove that $f(Tx) = 0$. But then $f(Tx) = (T^*f)(x) = 0$ since $T^*f = 0$.

Problem 4.7.7. This problem just states the contrapositive[4] of the uniform boundedness theorem. So there really is nothing to do here. (But it is useful to have the theorem in this form.)

Problem 4.7.8. Using the notation (almost)[5] introduced in the problem, if $x \in X$ with $x_j = 0$ when $j \ge J$, then $f_n(x) = 0$ if $n > J$, otherwise $|f_n(x)| = n\,|x_j| \le J\|x\|_\infty$. So the family $(f_n)_{n=1}^\infty$ is pointwise bounded. However, it is not uniformly bounded, for $\|f_n\| = n$.

Extra: *Prove that a closed subspace of a reflexive space is reflexive.*

Let X be a reflexive space and $Y \subseteq X$ a closed subspace. Assume $\eta \in Y^{**}$. Define $\xi \in X^{**}$ by setting

$$\xi(f) = \eta(f|_Y) \qquad (f \in X^*).$$

Since X is reflexive, the functional ξ is really of the form $f \mapsto f(x)$ for some $x \in X$. So the above definition becomes

$$\eta(f|_Y) = f(x) \qquad (f \in X^*).$$

We claim that $x \in Y$. For if $x \notin Y$, there is a bounded linear functional on X with $f|Y = 0$ and $f(x) \neq 0$ (because Y is closed, see Lemma 4.6-7). But this is impossible since then $0 \neq f(x) = \eta(f|_Y) = \eta(0) = 0$.

So we now write

$$\eta(g) = g(x) \qquad (g = f|_Y,\ f \in X^*).$$

But, by the Hahn–Banach theorem, *every* bounded linear functional on Y can be written $f|_Y$ with $f \in X^*$. Thus $\eta(g) = g(x)$ for all $g \in Y^*$, where $x \in Y$. This proves that Y is reflexive.

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[1] Recall that I write $M^\perp$ for the annihilator, where Kreyszig writes M^a.

[2] $\forall$ is short for "for all".

[3] What Kreyszig calls the annihilator ${}^aB$, I prefer to call the *preannihilator* and write as $B_\perp$ (the annihilator of a subset of X^* is contained in X^{**}).

[4] The *contrapositive* of a statement on the form "if A then B" is the equivalent statement "if not B then not A".

[5] I dislike the convention of using different letters for a vector and its components, as in $x = (\xi_j)$. There aren't enough letters in the alphabet, and this is wasteful.