

Exercise set A

Some exercises for TMA4230 Functional analysis

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Exercise A.1. (This is just a warmup exercise, in case these results are not familiar. Skip it if you don't feel you need to do it.) Whenever we work within a fixed set X , it is convenient to write the complement of a subset $A \subseteq X$ as $A^c = X \setminus A$. Prove that, when A and B are subsets of X and \mathcal{C} is a set of subsets of X , that (these are known as de Morgan's laws)

$$\begin{aligned} A^c \cap B^c &= (A \cup B)^c, & \bigcap \{C^c : C \in \mathcal{C}\} &= (\bigcup \mathcal{C})^c, \\ A^c \cup B^c &= (A \cap B)^c, & \bigcup \{C^c : C \in \mathcal{C}\} &= (\bigcap \mathcal{C})^c. \end{aligned}$$

Also, when $f: Y \rightarrow X$ then

$$\begin{aligned} f^{-1}(A \cap B) &= f^{-1}(A) \cap f^{-1}(B), & f^{-1}(\bigcap \mathcal{C}) &= \bigcap \{f^{-1}(C) : C \in \mathcal{C}\}, \\ f^{-1}(A \cup B) &= f^{-1}(A) \cup f^{-1}(B), & f^{-1}(\bigcup \mathcal{C}) &= \bigcup \{f^{-1}(C) : C \in \mathcal{C}\}, \\ f^{-1}(A \setminus B) &= f^{-1}(A) \setminus f^{-1}(B). \end{aligned}$$

Exercise A.2. Let X and Y be topological spaces. Show that a function $f: X \rightarrow Y$ is continuous if and only if $f^{-1}(F)$ is closed in X for every closed $F \subseteq Y$.

Exercise A.3. We have defined compactness of X in terms of open covers of X , which are sets of open subsets of X covering X .

Instead, consider now a subset K of a topological space X . Then K with the topology inherited from X is a topological space in its own right, so we can ask if K is compact or not.

If we define an open cover of K to be a set of open subsets of X whose union contains K , prove that K is compact if and only if every open cover of K has a finite subcover (of K). (The point here is that compactness of K is defined in terms of open covers of K , as consisting of subsets of K which are open in the inherited topology.)

Exercise A.4. Show that any closed subset of a compact space is compact.

Exercise A.5. Show that any compact subset of a Hausdorff space is closed.

Exercise A.6. Let X be a compact Hausdorff space.

(a) Show that if you replace the topology on X by a strictly stronger topology, then in the new topology X is still Hausdorff but no longer compact.

(b) Show that if you replace the topology on X by a strictly weaker topology, then in the new topology X is still compact but no longer Hausdorff.

Hint. Show both the easy parts first (Hausdorff in (a), compact in (b)). For (a) and (b) both, assume F is closed in the stronger topology but not in the weaker topology, and consider the compactness (or not) of F in the two topologies to arrive at a contradiction.

Exercise A.7. Let X be a set and \mathcal{B} a set of subsets of X . Let \mathcal{B}' be the set of all finite intersections from \mathcal{B} :

$$\mathcal{B}' = \{B_1 \cap \cdots \cap B_n : B_1, \dots, B_n \in \mathcal{B}; n = 0, 1, 2, \dots\}$$

with the understanding that $B_1 \cap \cdots \cap B_n = X$ when $n = 0$. Let \mathcal{T} consist of all possible unions of members of \mathcal{B}' . Show that \mathcal{T} is a topology; in fact, it is the weakest topology containing \mathcal{B} . It is said to be the topology *generated* by \mathcal{B} . Also, \mathcal{B} is said to be a *basis* for \mathcal{T} .

Exercise A.8. A topology is called *second countable* if it has a countable basis. It is called *first countable* if every neighbourhood filter has a countable filter base. Show that any metrizable space is first countable. Also show that any second countable space is first countable. Give an example of a first countable space that is not second countable.