Equilibria for planar systems

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This note is about the classification of equilibrium points of nonlinear systems in the plane. We shall deal only with those cases which can be (mostly) decided on the basis of the linearization of the systems around the equilibrium.

But first, a few generalities that apply in any dimension: Consider a system of the form

$$\dot{z} = f(z)$$

where the unknown function z is vector valued: $z(t) \in \mathbb{R}^n$.

If z_0 is an equilibrium point, i.e., if $f(z_0) = 0$, then the change of variables $w = z - z_0$ changes the system into one with an equilibrium at 0. So we lose no generality in assuming that $z_0 = 0$, and shall do so through this note.

So assume now that $f(0) = 0$, and also that f is *differentiable* at 0: Thus

$$f(z) = Az + o(r), \qquad z \to 0,$$

where A is an $n \times n$ matrix, called the *Jacobian matrix* or the *derivative* of f at 0.[1] We write $df(0)$ or even $f'(0)$ for this matrix.

In order to see what information can be gained from the Jacobian matrix, it is useful to reduce attention to a handful of *normal forms*. If we perform a linear change of variables:

$$z = Vw,$$

where V is an invertible matrix, the system is transformed into the form $V\dot{w} = f(Vw)$, or

$$\dot{w} = g(w), \qquad \text{where } g(w) = V^{-1}f(Vw).$$

Since $g(w) = V^{-1}f(Vw) = V^{-1}\big(f'(0)Vw + o(|Vw|)\big) = V^{-1}f'(0)Vw + o(|w|)$, we must have

$$g'(0) = V^{-1}f'(0)V.$$

In other words, $g'(0)$ is *similar* to $f'(0)$.

[1] Actually, I prefer to think of it as a linear functional.

Normal forms of equilibria for planar systems

It is a fundamental result of linear algebra that every real 2×2 matrix is similar one of the following *normal forms*:

Two distinct, real eigenvalues:

$$\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$$

These give rise to the *nodes* ($\lambda_1\lambda_2 > 0$) and *saddle points* ($\lambda_1\lambda_2 < 0$). The degenerate cases ($\lambda_1\lambda_2 = 0$) require more detailed analysis and will be skipped here.

Two complex eigenvalues $\sigma \pm i\omega$, with $\sigma,\omega \in \mathbb{R}$ and $\omega \neq 0$:

$$\begin{pmatrix}\sigma & -\omega\\ \omega & \sigma\end{pmatrix}$$

These give rise to *foci* [singular: *focus*] when $\sigma \neq 0$. The cases where $\sigma = 0$ (the linearization is a center) require more detailed analysis

One real eigenvalue (two possibilities):

$$\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix} \qquad \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$$

These are borderline cases between the nodes and the foci.

Nodes

We consider a planar system of the form

$$\begin{aligned}\dot{x} &= \lambda_1 x + g(x,y), & g(x,y) &= o(r),\\ \dot{y} &= \lambda_2 y + h(x,y), & h(x,y) &= o(r),\end{aligned}$$

where the "little-oh" notation refers to the limit as $r = \sqrt{x^2+y^2} \to 0$. We must also assume that g and h are sufficiently regular that the existence and uniqueness theorems hold: C^1 is the usual assumption, but Lipschitz continuity is sufficient in the first part of the analysis.

To investigate stability of the system, use polar coordinates:

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \frac{\lambda_1 x^2 + \lambda_2 y^2 + xg(x,y) + yh(x,y)}{r} = \frac{\lambda_1 x^2 + \lambda_2 y^2}{r} + o(r).$$

First, assuming $\lambda_1 \le \lambda_2 < 0$ we conclude

$$\dot r < \lambda_2 r + o(r) < (\lambda_2 + \varepsilon) r$$

for any ε and r small enough.[2] Thus $r \to 0$ exponentially as $t \to \infty$, if the starting value is small enough. This is the *stable* case.

But we can say more: We also find

$$\dot\theta = \frac{x\dot y - y\dot x}{r^2} = \frac{\lambda_2 x\bigl(y + h(x,y)\bigr) - \lambda_1 y\bigl(x + g(x,y)\bigr)}{r^2} = (\lambda_2 - \lambda_1)\frac{xy}{r^2} + o(1),$$

and using $xy = r\cos\theta \cdot r\sin\theta = \frac{1}{2} r \sin 2\theta$,

$$\dot\theta = \tfrac{1}{2}(\lambda_2 - \lambda_1)\sin 2\theta + o(1).$$

If r is small enough, the final $o(1)$ term will be small enough so that $\dot\theta$ must have the same sign as $\sin 2\theta$, except inside four narrow sectors around the axes, as indicated in figure 1.

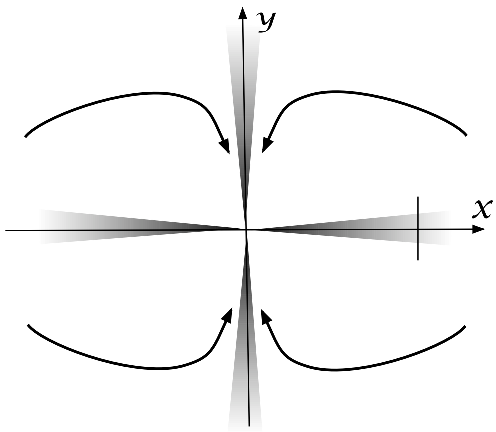

Figure 1: An attractive node

Outside the shaded sectors, the movement must be roughly as indicated, in the sense that $\dot r < 0$ and the sign of $\dot\theta$ must be as shown by the arrows.

[2] The precise statement: Given any $\varepsilon > 0$ there is some $\delta > 0$ so that the inequality holds whenever $r < \delta$.

In particular, any trajectory that does not stay within the horizontal sectors must end up inside the vertical ones. (For example, looking in the first quadrant but outside the shaded sectors, we have a definite lower positive bound on $\dot{\theta}$, so the trajectory must cross into the sector around the y axis in a finite time.)

By considering ever small scales, we can redraw the figure with ever narrower shaded sectors. So we conclude that the majority of trajectories will approach the origin from the vertical direction, but it is conceivable that some trajectories will do so horizontally.

In fact, some of them must do so, as we show next.

Consider initial data (x_0, y_0) with $x_0 > 0$ fixed and small, while y_0 varies (i.e., on the thin vertical line indicated in the righthand side of Figure 1). For large enough (but still small) $|y_0|$, the initial point will be outside the shaded sector, and so the solution will remain outside the sector. For others, the solution will escape the sector either upwards or downwards. Now, by the continuous dependence of solutions with respect to initial data, the set of y_0 for which the solution escapes upwards will be an open set A, and the set for which it escapes downwards will be another open set B. In fact, these sets will be intervals, since solution curves cannot cross, so any solution trapped above one that escapes upwards will itself do so, and similarly for the downward escaping ones. So there must be at least one y_0 that belongs to neither A nor B, and the trajectory through this point must approach the origin while staying inside the sector (since there is nowhere else for it to go).

If we assume a bit more regularity of the righthand side of the equation, one can show that only one curve from each side will approach the origin horizontally, but we shall not prove it here.

If both eigenvalues are positive rather than negative, we get the described behaviour of solutions, but this time as $t \to -\infty$. There is no need to repeat the analysis: Just apply the results of the above reasoning to the reversed system.

Coinciding eigenvalues.

$$\begin{aligned} \dot{x} &= \lambda x + \varepsilon y + g(x, y), & g(x, y) &= o(r), \\ \dot{y} &= \lambda y + h(x, y), & h(x, y) &= o(r), \end{aligned}$$

with $\lambda \neq 0$. The case $\varepsilon = 0$ is the case where the eigenspace corresponding to the single eigenvalue is two-dimensional, while the case $\varepsilon \neq 0$ corresponds to a Jordan normal form with a one-dimensional eigenspace. The latter is usually specified with $\varepsilon = 1$, but replacing y by εy yields the above form.

We shall insist on having $0 \le \varepsilon < |\lambda|$: For then

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \frac{\lambda r^2 + \varepsilon xy}{r} + o(r).$$

Noting as before that $xy = \frac{1}{2}r^2 \sin 2\theta$, we find $|\varepsilon xy| \le \frac{1}{2}r^2 < \frac{1}{2}|\lambda|$, so that $\dot{r}$ has the same sign as λ when r is small enough, and solutions tend to 0 exponentially as $t \to \infty$ (if $\lambda < 0$) or $t \to -\infty$ (if $\lambda > 0$).

In other words, the question of stability is settled just as previously. What happens in the angular (θ) direction is far less clear-cut.

Saddles

Next, we consider the case

$$\begin{aligned} \dot{x} &= -\lambda x + g(x, y), \qquad & g(x, y) &= o(r), \\ \dot{y} &= \mu y + h(x, y), \qquad & h(x, y) &= o(r), \end{aligned}$$

where $\lambda, \mu > 0$ and the "little-oh" notation again refers to the limit as $r = \sqrt{x^2 + y^2} \to 0(0, 0)$.

Polar coordinates are not quite as useful in this case, but we can instead note that $\dot{x}$ and x have opposite signs, so long as r is small and we stay outside the thin sectors around the y axis in Figure 2.

In detail: Let $\varepsilon > 0$. If r is small enough then $|g(x, y)| < \varepsilon r$. Whenever $\varepsilon r < \lambda|x|$, then $\dot{x}$ and x must indeed have opposite signs. Squaring the inequality we get $\varepsilon^2(x^2 + y^2) < \lambda^2 x^2$, and assuming $\varepsilon < \lambda$, that is true if $\sqrt{\lambda^2 - \varepsilon^2}\,|x| < \varepsilon|y|$.

A similar analysis shows that $\dot{y}$ and y have the same signs outside the small sectors around the x axis.

So outside the four sectors, all solutions must move in the general direction indicated by the arrows.

In particular, any solution that strays outside the two horizontal sectors will escape out of the neighbourhood in the vertical direction.

Repeating the argument from the node case, considering initial data on the small vertical line segment across the positive x axis, we find that some trajectory will approach the origin horizontally from the right (and similarly, one from the left).

Once more, assuming a bit more regularity of the system we can show that there is only one such trajectory from each side.

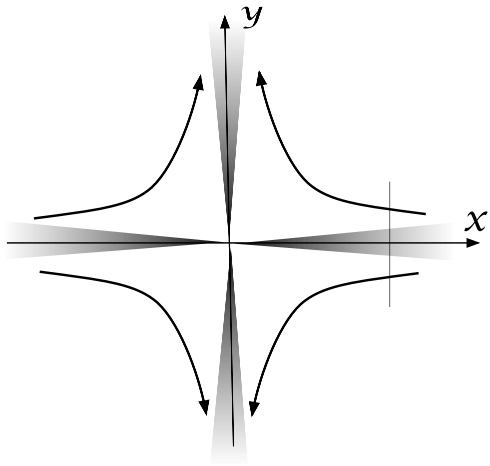

Figure 2: A saddle point

This is somewhat easier to see than the corresponding result for nodes: Assume the righthand side of the system is C^1. Thus g and h are C^1 functions. Since both are $o(r)$, the first order partial derivatives of each are 0 at $(0,0)$, and it follows that $\partial_y h = \partial h/\partial y = o(r)$ as $r \to 0$ (and similarly with the three other derivatives). In particular, consider trajectories through two nearby points (x, y_1) and (x, y_2) and compare $\dot{y}$ at the two points:

$$\mu y_2 + h(x, y_2) - \mu y_1 - h(x, y_1) = \mu(y_2 - y_1) + o(r) \qquad (r \to 0),$$

since the secant theorem implies $g(x, y_2) - g(x, y_1) = (y_2 - y_1)\partial_y h(x, \eta)$ for some η between y_1 and y_2. So we get $\dot{y}_2 - \dot{y}_1 \approx \mu(y_2 - y_1)$, and the two trajectories must move apart exponentially in the vertical direction. (This analysis is not quite rigorous, since the two solutions will also move apart in the x direction, albeit more slowly. But this minor problem can be fixed.)

Taken together with the origin itself, the two curves approaching the origin from each side form what is known as the *stable curve* of the equilibrium point. This curve is tangential with the x axis. All initial data near the equilibrium point and not on the stable curve, must escape out of small neighbourhoods.

Now we can reverse time and repeat the argument. In the original system, we conclude that there is an *unstable curve* which is tangent to the y

axis, along which solutions tend to the equilibrium points as $t \to -\infty$. And all other initial data produces solutions that escape the neighbourhood when time runs backwards.

Foci

Finally, we have saved the easiest case for last: systems on the form

$$\dot{x} = \sigma x - \omega y + g(x,y), \qquad g(x,y) = o(r),$$
$$\dot{y} = \omega x + \sigma y + h(x,y), \qquad h(x,y) = o(r),$$

where $\omega \neq 0$. In polar coordinates we get

$$\dot{r} = \sigma r + o(r), \quad \dot{\theta} = \omega + o(1), \qquad \text{as } r \to 0.$$

The first equation shows that if $\sigma < 0$ then $r \to 0$ exponentially fast as $t \to \infty$, and the equilibrium is *stable*.

Similarly, if $\sigma > 0$ then $r \to 0$ exponentially fast as $t \to -\infty$, and the equilibrium is *unstable*.

In either case, θ grows approximately at a linear rate as $r \to 0$, so the solution spirals around the equilibrium point an infinite number of times.

This behaviour defines a *focus* in general.

Note that in the degenerate case $\sigma = 0$ the linearized system is a center, but anything might happen to the nonlinear system: It could be a center or a focus, or there could be an infinite sequence of closed trajectories around the origin, typically with spirals in between.

Summary

Via a linear change of coordinates the above analysis applies to equilibrium points of any C^1 planar system, so long as the eigenvalues of the Jacobian matrix are distinct and have nonzero real part.

If both eigenvalues are real and of the same sign, we get a *node*. By definition, a node is either stable or unstable: In the stable case, all nearby trajectories approach the equilibrium as $t \to \infty$. In the ustable case, they approach the equilibrium as $t \to -\infty$. And in either case, they all do so tangentially to a common line through the equilibrium point, with just two exceptions, which approach from opposite sides tangentially to a different line. The two lines are parallel to the eigenspaces of the Jacobian matrix.

The node is stable if the eigenvalues are negative, and unstable if they are positive. This result holds even if the eigenvalues are equal, but in the latter case behaviour may be like either a node or a focus, and more detailed analysis is needed.

If both eigenvalues are real and of opposite signs, we get a *saddle point*. Through the saddle point are two curves, the *stable curve* which is tangent to the eigenspace corresponding to the negative eigenvalue, and the *unstable curve* which is tangent to the eigenspace corresponding to the positive eigenvalue. The stable and unstable curves are each composed of two trajectories and the equilibrium point itself. The two trajectories on the stable curve approach the equilibrium point as $t \to \infty$, and those on the unstable curve approach the equilibrium point as $t \to -\infty$.

Finally, in the case of non-real eigenvalues, the two eigenvalues will be complex conjugates of each other, and we get a *focus*: Solutions will spiral towards the equilibrium either as $t \to \infty$ (the stable case) or $t \to -\infty$ (the unstable case).