The Poincaré–Bendixson theorem

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The Poincaré–Bendixson theorem is often misstated in the literature. The purpose of this note is to try to set the record straight, and to provide the outline of a proof.

Throughout this note we are considering an autonomous dynamical system on the form

$$\dot{x} = f(x), \qquad x(t) \in \Omega \subseteq \mathbb{R}^n$$

where $f\colon \Omega \to \mathbb{R}^n$ is a locally Lipschitz continuous vector field on the open set Ω.

Furthermore, we are considering a solution x whose *forward half orbit* $O_+ = \{x(t)\colon t \ge 0\}$ is contained in a compact set $K \subset \Omega$.

An *omega point* of O_+ is a point z so that one can find $t_n \to +\infty$ with $x(t_n) \to z$. It is a consequence of the compactness of K that omega points exist. Write ω for the set of all omega points of O_+.

It should be clear that ω is a closed subset of K, and therefore compact. Also, as a consequence of the continuous dependence of initial data and the general nature of solutions of autonomous systems, ω is an invariant set (both forward and backward) of the dynamical system.

We can now state our version of the main theorem.

1 Theorem. (Poincaré–Bendixson) *Under the above assumptions, if ω does not contain any equilibrium points, then ω is a cycle. Furthermore, either the given solution x traverses the cycle ω, or it approaches ω as $t \to +\infty$.*

What happens if ω does contain an equilibrium point?

The simplest case is the case $\omega = \{x_0\}$ for an equilibrium point x_0. Then it is not hard to show that $x(t) \to x_0$ as $t \to +\infty$. (If not, there is some $\varepsilon > 0$ so that $|x(t) - x_0| \ge \varepsilon$ for arbitrarily large t, but then compactness guarantees the existence of another omega point in $\{z \in K\colon |z - x_0| \ge \varepsilon\}$.)

I said in the introduction that the Poincaré–Bendixson theorem is often misstated. The problem is that the above two possibilities are claimed to be the only possibilities. But a third possibility exists: ω can consist of one or more equilibrium points joined by solution paths starting and ending at these equilibrium points (i.e., heteroclinic or homoclinic orbits).

2 Example. Consider the dynamical system

$$\left.\begin{aligned}\dot{x} &= \frac{\partial H}{\partial y} + \mu H \frac{\partial H}{\partial x}\\ \dot{y} &= -\frac{\partial H}{\partial x} + \mu H \frac{\partial H}{\partial y}\end{aligned}\right\}, \qquad H(x,y) = \tfrac{1}{2}y^2 - \tfrac{1}{2}x^2 + \tfrac{1}{4}x^4.$$

Notice that if we set the parameter μ to zero, this is a Hamiltonian system. Of particular interest is the set given by $H = 0$, which consists of the equilibrium point at zero and two homoclinic paths starting and ending at this equilibrium, roughly forming an ∞ sign.

In general, an easy calculation gives

$$\dot{H} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = \mu H \cdot \left[\left(\frac{\partial H}{\partial x}\right)^2 + \left(\frac{\partial H}{\partial y}\right)^2\right]$$

so that H will tend towards 0 if $\mu < 0$. In particular, any orbit starting outside the "∞ sign" will approach it from the outside, and the "∞ sign" itself will be the omega set of this orbit.

Figure 1 shows a phase portrait for $\mu = -0.02$.

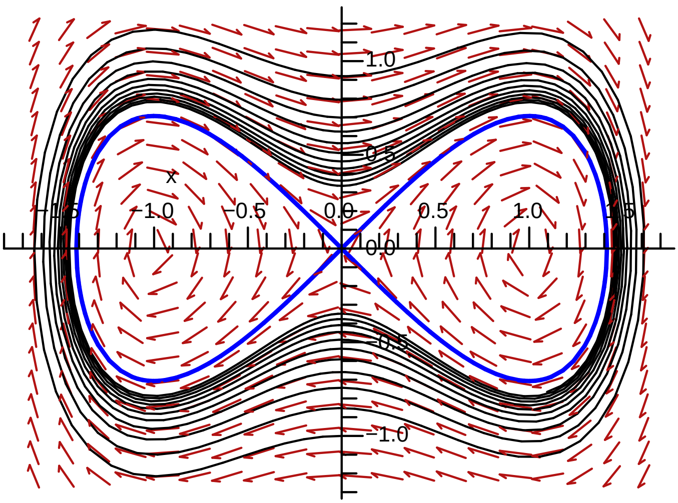

Figure 1: An orbit and its omega set.

We now turn to the proof of theorem 1.

By a *transverse line segment* we mean a closed line segment contained in Ω, so that f is not parallel to the line segment at any point of the segment. Thus the vector field points consistently to one side of the segment.

Clearly, any non-equilibrium point of Ω is in the interior of some transverse line segment.

3 Lemma. *If an orbit crosses a transverse line segment L in at least two different points, the orbit is not closed. Furthermore, if it crosses L several times, the crossing points are ordered along L in the same way as on the orbit itself.*

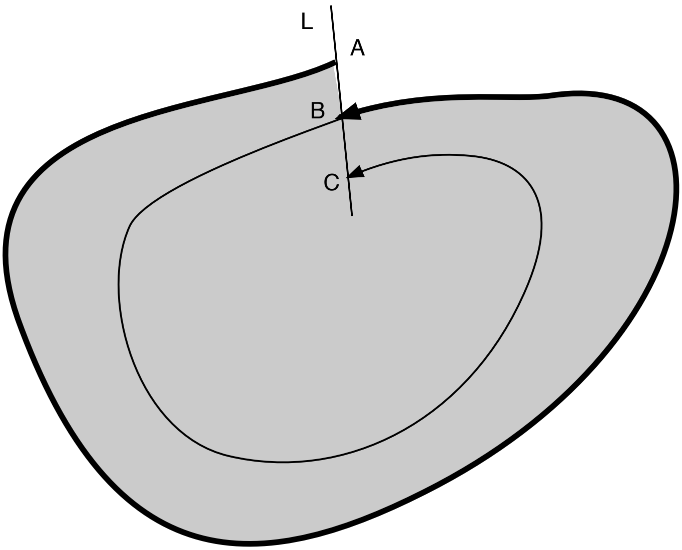

Figure 2: Crossings of a transverse line segment

Proof: Figure 2 shows a transverse line segment L and an orbit that crosses L, first at A, then at B. Note that the boundary of the shaded area consists of part of the orbit, which is of course not crossed by any other orbit, and a piece of the L, at which the flow enters the shaded region. (If B were to the other side of A, we would need to consider the outside, not the inside, of the curve.) In particular, there is no way the given orbit can ever return to A. Thus the orbit is not closed.

It cannot return to any other point on L between A and B either, so if it ever crosses L again, it will have to be further along in the same direction on L, as in the point C indicated in the figure. (Hopefully, this clarifies the somewhat vague statement at the end of the lemma.) ∎

4 Corollary. *A point on some orbit is an omega point of that orbit if, and only if, the orbit is closed.*

Proof: The "if" part is obvious. For the "only if" part, assume that A is a point that is also an omega point of the orbit through A. If A is an equilibrium point, we have a special case of a closed orbit, and nothing more to prove. Otherwise, draw a transverse line L through A. Since A is also an omega point, some future point on the orbit through A will pass sufficiently close to A that it must cross L at some point B. If the orbit is not closed then $A \neq B$, but then any future point on the orbit is barred from entering a neighbourhood of A (consult Figure 2 again), which therefore cannot be an omega point after all. This contradiction concludes the proof. ∎

Outline of the proof of Theorem 1 Fix some $x_0 \in \omega$, and a transverse line segment L with x_0 in its interior.

If x_0 happens to lie on O_+ the corollary above shows that the orbit through x_0 must be closed, so ω in fact equals that orbit.

If x_0 does not lie on O_+ then O_+ is not closed. However, I claim that the orbit through x_0 is still closed. In fact, let z_0 be an omega point of the orbit through x_0, and draw a transverse line L through z_0. If the orbit through x_0 is not closed, it must pass close enough to z_0 that it must cross L, infinitely often in a sequence that approaches z_0 from one side. In particular, it crosses at least twice, say, first at A and then again at B (again, refer to Figure 2).

But B is an omega point of O_+, so O_+ crosses L arbitrarily close to B, and so O_+ enters the shaded area in the figure. But then it can never again get close to A. This is a contradiction, since A is also an omega point of O_+.

We have shown that x_0 lies on a closed path. This closed path must be all of ω. The solution x gets closer and closer to ω, since it crosses a transverse line segment through x_0 in a sequence of points approaching x_0, and the theorem on continuous dependence on initial data does the rest. ∎