A two-fluid four-equation model with instantaneous thermodynamical equilibrium

ALEXANDRE MORIN
Department of Energy and Process Engineering,
Norwegian University of Science and Technology (NTNU),
Kolbjørn Hejes vei 1B, NO-7491 Trondheim, Norway.
alexandre.morin [a] sintef.no

AND TORE FLÅTTEN
SINTEF Energy Research,
P.O. Box 4761 Sluppen, NO-7465 Trondheim, Norway

February 1, 2013

Abstract

We analyse a four-equation version of a common two-fluid model for pipe flow, containing one mixture mass equation and one mixture energy equation. The motivation is to obtain a fluid-dynamical model where the mixture is in thermodynamical equilibrium at all time. We start from a five-equation model with instantaneous thermal equilibrium, to which we add phase relaxation terms. An interfacial velocity appears, for which we give an expression based on the second law of thermodynamics. We then derive the limit of this model when the relaxation becomes instantaneous. The time derivatives appearing in this process are subsequently transformed into spatial derivatives to be able to use numerical methods for conservation laws. The Jacobian matrix of the fluxes can then be evaluated, and the system be put into quasilinear form. From the Jacobian matrix, we are able to extract the sound speed intrinsic to the model. By comparison to the sound speed in other two-phase flow models, we extend some previous results showing that the effect of relaxation on sound speed is independent of the order in which the variables are relaxed. We also check the subcharacteristic condition and place the model in a hierarchy of two-phase flow models. Finally, this model requires a regularisation term to be hyperbolic. With the help of a perturbation method, we find an expression for this term that makes the model conditionally hyperbolic. Two-phase flows, relaxation, two-fluid model, subcharacteristic condition

1 Introduction

One-dimensional two-phase flows in pipelines may be modelled using the two-fluid model ((Munkejord *et al.* 2009, Paillère *et al.* 2003, Stewart & Wendroff 1984, Toumi 1996)). The two-fluid model is characterised by the fact that it has two momentum equations. Therefore, the phase velocities are independent from each other, as opposed to the drift-flux model ((Flåtten *et al.* 2010, Murrone & Guillard 2005, Saurel *et al.* 2008)) where there is only one momentum equation for the mixture. The six-equation version of the two-fluid model is used for example in the nuclear industry ((Bestion 1990, WAHA3 Code Manual 2004)). In this version, the phases are in mechanical equilibrium – they are at the same pressure at all time – but not in chemical and thermal equilibrium. A five-equation version has been chosen for pipeline flow simulation ((Bendiksen *et al.* 1991)), in which the phases are assumed to be in mechanical and thermal equilibrium. A seven-equation version, where the phases are allowed to be totally out of equilibrium – both have their own pressure, temperature and chemical potential – has also been derived ((Baer & Nunziato 1986, Saurel & Abgrall 1999)). One quality of the latter model is that it avoids the non-hyperbolicity ((Gidaspow 1974, Stuhmiller 1977)) of the six-equation model.

Relaxation source terms may be added to the model to bring it towards equilibrium at a finite rate. This has been studied for example by (Martínez Ferrer *et al.* 2012, Flåtten & Lund 2011, Karlsen *et al.* 2004, Natalini 1999, Pareschi & G. Russo 2005, Saurel & Abgrall 1999, Tran *et al.* 2009). An equilibrium system may also be approached by a relaxation system with very stiff source terms ((Aursand *et al.* 2011)). For instance, the six-equation model with a stiff temperature relaxation will behave similarly to the five-equation model with one mixture energy equation. However, numerical methods for hyperbolic systems do not naturally handle algebraic source terms.

Table 1: Main symbols.

Symbol	Signification	Symbol	Signification
c	Speed of sound	Γ	First Grüneisen coefficient
C_p	Specific heat capacity at constant pressure	ε	Perturbation parameter
e	Internal energy	μ	Chemical potential
E	Phasic total energy ($E = \alpha\rho(e + 1/2v^2)$)	ρ	Density
f_i	Components of the vector F	A	Jacobian
p	Pressure	B	Coefficient matrix in the non-conservative terms
T	Temperature	F	Vector of the fluxes
u_i	Components of the vector U	U	Vector of the conserved variables
v	Velocity	W	Vector of the non-conservative variables
w_i	Components of the vector W	g	Gas phase (Subscript)
α	Volume fraction	ℓ	Liquid phase (Subscript)

With a splitting approach, the fluxes are advanced one time step alternately with the source terms. The latter are solved using ordinary differential equation solvers. However, when the relaxation is instantaneous, it should directly affect the propagation speed of the waves. This time splitting may cause smearing of the discontinuities in this case. Thus, it is preferable to use the equilibrium system.

For the simulation of the two-phase flow of a mixture with phase change, the equation of state plays an important role. For example, the Span-Wagner equation of state is accurate for two-phase mixtures of CO_2 ((Span & Wagner 1996)). However it is an equilibrium equation of state, which means that the fluid-dynamical model must handle a mixture that is at equilibrium at all time. Therefore, a four-equation version of the two-fluid model has to be derived in order to use such equilibrium-based equations of state. This model was mentioned by (Schor *et al.* 1984). However, the treatment of the momentum-exchange terms due to phase change was not mentioned. These terms require a careful treatment, because phase change becomes instantaneous. In the present paper, we derive the four-equation model from the two-fluid five-equation model presented by (Martínez Ferrer *et al.* 2012), where we replace the individual phase mass-equations by a mixture mass equation and an instantaneous chemical equilibrium assumption. As mentioned in the previous paragraph, this will modify the wave structure of the model compared to the initial five-equation model. In fact, this phenomenon has been studied, and a stability condition has been derived, called the subcharacteristic condition ((Chen *et al.* 1994, Martínez Ferrer *et al.* 2012, Flåtten & Lund 2011, Liu 1987a, Natalini 1999)). It says that for a relaxation system and its corresponding equilibrium system, the speed of the waves of corresponding families will be lower in the equilibrium system than in the relaxation system. (Martínez Ferrer *et al.* 2012) began to establish a hierarchy of two-phase flow models with respect to the subcharacteristic condition, where they concentrated on velocity and thermal relaxation. In addition, they showed that the sound speed is reduced by the same factor regardless of the order in which the relaxation processes are performed.

The four-equation model thus derived is expected to be non-hyperbolic when the gas and liquid velocities are different from each other. Therefore, we add to the derivation a regularising term. We choose to use an interfacial pressure term of the sort often used with the six-equation two-fluid model ((Bestion 1990, Coquel *et al.* 1997, Cortes *et al.* 1998, Evje & Flåtten 2003, Paillère *et al.* 2003, Toumi 1996)). We then obtain an explicit expression for the pressure difference involved in this term. We do this with the help of a perturbation method ((Toumi & Kumbaro 1996, Toumi 1996)). It is interesting to remark that this term is identical to a well-known form for the six-equation model ((Chang & Liou 2007, Evje & Flåtten 2003, Munkejord 2007, Munkejord *et al.* 2009, Paillère *et al.* 2003, Stuhmiller 1977)).

The structure of the paper is as follows. In Section 2, we present the five-equation model, to which we add relaxation source terms for phase change. These involve an interfacial momentum velocity, for which we derive a precise expression with the help of entropy considerations. In Section 3, the four-equation model is analysed. The phase change relaxation source term is expressed by means of derivatives, so that no algebraic terms remain in the system. Also, the problematic time derivatives are transformed into spatial derivatives. Then, in Section 4, the system is written in quasilinear form, which involves finding the Jacobian of the fluxes. In Section 5, the speed of sound of the model is evaluated, and the subcharacteristic condition with respect to other two-phase flow models verified. A main result of the present paper is the equation (5.18), which extends previous results on the effect of relaxation on the speed of sound. In Section 6, we show how a perturbation method gives an expression for the interfacial pressure difference that makes the model hyperbolic. Finally, in Section 7, we discuss the phenomenon of resonance which is known to occur in the kind of two-fluid models we consider ((Isaacson & Temple 1990, Liu 1987b, Morin *et al.* 2012)). Section 8 summarises the results of the paper. The main symbols used are listed in Table 1. The other ones are introduced in the text.

2 The five equation model with phase relaxation

The two-fluid five-equation model studied by (Martínez Ferrer *et al.* 2012) describes a one-dimensional two-phase flow where the pressure and the temperature are kept equal in both phases at all times. This follows from the assumption of instantaneous mechanical and thermal equilibrium. However, the two phases will in general not be in chemical equilibrium. Algebraic relaxation terms representing phase change should then act to attract the phases towards equilibrium. After addition of phase relaxation, the five-equation model becomes

$$\frac{\partial \alpha_{\mathrm{g}}\rho_{\mathrm{g}}}{\partial t}+\frac{\partial \alpha_{\mathrm{g}}\rho_{\mathrm{g}}v_{\mathrm{g}}}{\partial x}=\mathscr{K}(\mu_\ell-\mu_{\mathrm{g}}), \tag{2.1}$$

$$\frac{\partial \alpha_\ell\rho_\ell}{\partial t}+\frac{\partial \alpha_\ell\rho_\ell v_\ell}{\partial x}=\mathscr{K}(\mu_{\mathrm{g}}-\mu_\ell), \tag{2.2}$$

$$\frac{\partial \alpha_{\mathrm{g}}\rho_{\mathrm{g}}v_{\mathrm{g}}}{\partial t}+\frac{\partial \alpha_{\mathrm{g}}\rho_{\mathrm{g}}v_{\mathrm{g}}^2}{\partial x}+\alpha_{\mathrm{g}}\frac{\partial p}{\partial x}=v_{\mathrm{i}}\mathscr{K}(\mu_\ell-\mu_{\mathrm{g}}), \tag{2.3}$$

$$\frac{\partial \alpha_\ell\rho_\ell v_\ell}{\partial t}+\frac{\partial \alpha_\ell\rho_\ell v_\ell^2}{\partial x}+\alpha_\ell\frac{\partial p}{\partial x}=v_{\mathrm{i}}\mathscr{K}(\mu_{\mathrm{g}}-\mu_\ell), \tag{2.4}$$

$$\frac{\partial (E_{\mathrm{g}}+E_\ell)}{\partial t}+\frac{\partial}{\partial x}((E_{\mathrm{g}}+\alpha_{\mathrm{g}}p)v_{\mathrm{g}}+(E_\ell+\alpha_\ell p)v_\ell)=0, \tag{2.5}$$

where

$$E=\alpha\rho\left(e+\frac{1}{2}v^2\right), \tag{2.6}$$

$\mathscr{K}$ is a positive relaxation constant, μ is the chemical potential, and v_{i} is some interface velocity. Assuming that the phases are composed of only one component, we may express the chemical potential as

$$\mu=e+\frac{p}{\rho}-Ts. \tag{2.7}$$

2.1 Interfacial momentum velocity

Through entropy considerations, we are able to give an expression for the interface velocity v_{i}.

Proposition 1. *If we assume that the interface velocity v_{i} is independent of $\mu_{\mathrm{g}}-\mu_\ell$, the second law of thermodynamics uniquely determines*

$$v_{\mathrm{i}}=\frac{1}{2}(v_{\mathrm{g}}+v_\ell). \tag{2.8}$$

Proof. We will derive the mixture entropy evolution equation, and impose that the source term should be non-negative. We first derive the kinetic energy evolution equations, by multiplying the momentum equations (2.3) and (2.4) by v_{g} and v_ℓ respectively. For the gas phase, after expansion of the derivatives, we obtain

$$v_{\mathrm{g}}^2\frac{\partial \alpha_{\mathrm{g}}\rho_{\mathrm{g}}}{\partial t}+\alpha_{\mathrm{g}}\rho_{\mathrm{g}}v_{\mathrm{g}}\frac{\partial v_{\mathrm{g}}}{\partial t}+v_{\mathrm{g}}^2\frac{\partial \alpha_{\mathrm{g}}\rho_{\mathrm{g}}v_{\mathrm{g}}}{\partial x}+\alpha_{\mathrm{g}}\rho_{\mathrm{g}}v_{\mathrm{g}}^2\frac{\partial v_{\mathrm{g}}}{\partial x}+\alpha_{\mathrm{g}}v_{\mathrm{g}}\frac{\partial p}{\partial x}=v_{\mathrm{g}}v_{\mathrm{i}}\mathscr{K}(\mu_\ell-\mu_{\mathrm{g}}). \tag{2.9}$$

The same applies to the liquid phase. After the use of the mass equation and reorganisation, the equations read

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\alpha_{\mathrm{g}}\rho_{\mathrm{g}}v_{\mathrm{g}}^2\right)+\frac{\partial}{\partial x}\left(\frac{1}{2}\alpha_{\mathrm{g}}\rho_{\mathrm{g}}v_{\mathrm{g}}^3\right)+\alpha_{\mathrm{g}}v_{\mathrm{g}}\frac{\partial p}{\partial x}=v_{\mathrm{g}}\left(v_{\mathrm{i}}-\frac{1}{2}v_{\mathrm{g}}\right)\mathscr{K}(\mu_\ell-\mu_{\mathrm{g}}), \tag{2.10}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\alpha_\ell\rho_\ell v_\ell^2\right)+\frac{\partial}{\partial x}\left(\frac{1}{2}\alpha_\ell\rho_\ell v_\ell^3\right)+\alpha_\ell v_\ell\frac{\partial p}{\partial x}=v_\ell\left(v_{\mathrm{i}}-\frac{1}{2}v_\ell\right)\mathscr{K}(\mu_{\mathrm{g}}-\mu_\ell). \tag{2.11}$$

Using the latter equations, we can now cancel the kinetic energy contribution in the mixture total energy equation (2.5), which gives

$$\begin{aligned}\frac{\partial}{\partial t}\left(\alpha_{\mathrm{g}}\rho_{\mathrm{g}}e_{\mathrm{g}}+\alpha_\ell\rho_\ell e_\ell\right)&+\frac{\partial}{\partial x}\left(\alpha_{\mathrm{g}}\rho_{\mathrm{g}}e_{\mathrm{g}}v_{\mathrm{g}}+\alpha_\ell\rho_\ell e_\ell v_\ell\right)\\&+p\frac{\partial \alpha_{\mathrm{g}}v_{\mathrm{g}}}{\partial x}+p\frac{\partial \alpha_\ell v_\ell}{\partial x}=(v_{\mathrm{g}}-v_\ell)\left(v_{\mathrm{i}}-\frac{1}{2}(v_{\mathrm{g}}+v_\ell)\right)\mathscr{K}(\mu_{\mathrm{g}}-\mu_\ell).\end{aligned} \tag{2.12}$$

By the mass equation, we obtain an evolution equation for the material derivatives of the phasic internal energy

$$\alpha_{\mathrm{g}}\rho_{\mathrm{g}}\frac{D_{\mathrm{g}}e_{\mathrm{g}}}{Dt}+\alpha_{\ell}\rho_{\ell}\frac{D_{\ell}e_{\ell}}{Dt}+p\frac{\partial\alpha_{\mathrm{g}}v_{\mathrm{g}}}{\partial x}+p\frac{\partial\alpha_{\ell}v_{\ell}}{\partial x} = \left((v_{\mathrm{g}}-v_{\ell})\left(v_{\mathrm{i}}-\frac{1}{2}(v_{\mathrm{g}}+v_{\ell})\right)+e_{\mathrm{g}}-e_{\ell}\right)\mathscr{K}(\mu_{\mathrm{g}}-\mu_{\ell}), \quad (2.13)$$

where we have introduced the phase specific material derivative $\frac{D_{\mathrm{k}}}{Dt}=\frac{\partial}{\partial t}+v_{\mathrm{k}}\frac{\partial}{\partial x}$.

Using the fundamental thermodynamic relation

$$\mathrm{d}e=\frac{p}{\rho^{2}}\,\mathrm{d}\rho+T\,\mathrm{d}s, \quad (2.14)$$

we can transform the previous equation into an entropy equation. First, (2.14) is expressed in terms of material derivatives and substituted in the internal energy equation (2.13)

$$\alpha_{\mathrm{g}}\rho_{\mathrm{g}}\left(T\frac{D_{\mathrm{g}}s_{\mathrm{g}}}{Dt}+\frac{p}{\rho_{\mathrm{g}}^{2}}\frac{D_{\mathrm{g}}\rho_{\mathrm{g}}}{Dt}\right)+\alpha_{\ell}\rho_{\ell}\left(T\frac{D_{\ell}s_{\ell}}{Dt}+\frac{p}{\rho_{\ell}^{2}}\frac{D_{\ell}\rho_{\ell}}{Dt}\right)+p\frac{\partial\alpha_{\mathrm{g}}v_{\mathrm{g}}}{\partial x}+p\frac{\partial\alpha_{\ell}v_{\ell}}{\partial x} = \left((v_{\mathrm{g}}-v_{\ell})\left(v_{\mathrm{i}}-\frac{1}{2}(v_{\mathrm{g}}+v_{\ell})\right)+e_{\mathrm{g}}-e_{\ell}\right)\mathscr{K}(\mu_{\mathrm{g}}-\mu_{\ell}). \quad (2.15)$$

By the mass equations (2.1)–(2.2), it can be simplified to

$$\alpha_{\mathrm{g}}\rho_{\mathrm{g}}T\frac{D_{\mathrm{g}}s_{\mathrm{g}}}{\partial t}+\alpha_{\ell}\rho_{\ell}T\frac{D_{\ell}s_{\ell}}{\partial t} = \left((v_{\mathrm{g}}-v_{\ell})\left(v_{\mathrm{i}}-\frac{1}{2}(v_{\mathrm{g}}+v_{\ell})\right)+e_{\mathrm{g}}+\frac{p}{\rho_{\mathrm{g}}}-e_{\ell}-\frac{p}{\rho_{\ell}}\right)\mathscr{K}(\mu_{\mathrm{g}}-\mu_{\ell}), \quad (2.16)$$

and using again the mass equations, we obtain the evolution equation for the mixture entropy

$$T\left(\frac{\partial\alpha_{\mathrm{g}}\rho_{\mathrm{g}}s_{\mathrm{g}}}{\partial t}+\frac{\partial\alpha_{\ell}\rho_{\ell}s_{\ell}}{\partial t}+\frac{\partial\alpha_{\mathrm{g}}\rho_{\mathrm{g}}s_{\mathrm{g}}v_{\mathrm{g}}}{\partial x}+\frac{\partial\alpha_{\ell}\rho_{\ell}s_{\ell}v_{\ell}}{\partial x}\right) = \left((v_{\mathrm{g}}-v_{\ell})\left(v_{\mathrm{i}}-\frac{1}{2}(v_{\mathrm{g}}+v_{\ell})\right)+\mu_{\mathrm{g}}-\mu_{\ell}\right)\mathscr{K}(\mu_{\mathrm{g}}-\mu_{\ell}) \quad (2.17)$$

since the chemical potential can be expressed as in (2.7). Let us name the right-hand side as

$$\mathscr{S}=\left((v_{\mathrm{g}}-v_{\ell})\left(v_{\mathrm{i}}-\frac{1}{2}(v_{\mathrm{g}}+v_{\ell})\right)+\mu_{\mathrm{g}}-\mu_{\ell}\right)\mathscr{K}(\mu_{\mathrm{g}}-\mu_{\ell}), \quad (2.18)$$

where we remind that $\mathscr{K}>0$. It may be written as

$$\mathscr{S}=\mathscr{K}\left(wz+z^{2}\right), \quad (2.19)$$

where

$$w=(v_{\mathrm{g}}-v_{\ell})\left(v_{\mathrm{i}}-\frac{1}{2}(v_{\mathrm{g}}+v_{\ell})\right), \quad (2.20)$$

$$z=\mu_{\mathrm{g}}-\mu_{\ell}. \quad (2.21)$$

Now, the second law of thermodynamics imposes

$$\mathscr{S}\geq 0. \quad (2.22)$$

For any given set of velocities, the entropy production attains its minimum when

$$\frac{\mathrm{d}\mathscr{S}}{\mathrm{d}z}=\mathscr{K}(w+2z)=0, \quad (2.23)$$

hence when

$$z=-\frac{w}{2}. \quad (2.24)$$

Inserting this into (2.19), we obtain

$$\mathscr{S} = -\mathscr{K}\frac{w^2}{4}. \tag{2.25}$$

Thus, the second law of thermodynamics imposes

$$w = 0, \tag{2.26}$$

which uniquely determines

$$v_{\mathrm{i}} = \frac{1}{2}(v_{\mathrm{g}} + v_\ell). \tag{2.27}$$

□

This is the same expression as proposed by (Stewart & Wendroff 1984), though it was not physically motivated.

3 The four-equation model

We wish to derive a four-equation model from the above five-equation model, where we assume the phase change to be instantaneous. The two phases will then at all times be in equilibrium. This is achieved by letting $\mathscr{K} \to \infty$ in the model (2.1)–(2.5). Since the repartition of the mass in the phases now is entirely governed by thermodynamics, we only need one mixture mass evolution equation, instead of one for each phase as in (2.1)–(2.2). We therefore sum (2.1) and (2.2) to give the mixture mass evolution equation of the four-equation model

$$\frac{\partial(\alpha_{\mathrm{g}}\rho_{\mathrm{g}} + \alpha_\ell\rho_\ell)}{\partial t} + \frac{\partial(\alpha_{\mathrm{g}}\rho_{\mathrm{g}}v_{\mathrm{g}} + \alpha_\ell\rho_\ell v_\ell)}{\partial x} = 0, \tag{3.1}$$

and specify $\mu_{\mathrm{g}} = \mu_\ell$. The remaining three other evolution equations of the four-equation model are the same as in the five-equation model (2.3)–(2.5). However, since $\mathscr{K} \to \infty$ and $\mu_{\mathrm{g}} = \mu_\ell$, $\mathscr{K}(\mu_{\mathrm{g}} - \mu_\ell)$ is an undefined limit. It needs to be substituted using the phase mass equations (2.1) and (2.2). This gives the model ((Schor *et al.* 1984))

$$\frac{\partial(\alpha_{\mathrm{g}}\rho_{\mathrm{g}} + \alpha_\ell\rho_\ell)}{\partial t} + \frac{\partial(\alpha_{\mathrm{g}}\rho_{\mathrm{g}}v_{\mathrm{g}} + \alpha_\ell\rho_\ell v_\ell)}{\partial x} = 0, \tag{3.2}$$

$$\frac{\partial \alpha_{\mathrm{g}}\rho_{\mathrm{g}}v_{\mathrm{g}}}{\partial t} + \frac{\partial \alpha_{\mathrm{g}}\rho_{\mathrm{g}}v_{\mathrm{g}}^2}{\partial x} + \alpha_{\mathrm{g}}\frac{\partial p}{\partial x} = \frac{v_{\mathrm{g}} + v_\ell}{2}\left(\frac{\partial \alpha_{\mathrm{g}}\rho_{\mathrm{g}}}{\partial t} + \frac{\partial \alpha_{\mathrm{g}}\rho_{\mathrm{g}}v_{\mathrm{g}}}{\partial x}\right), \tag{3.3}$$

$$\frac{\partial \alpha_\ell\rho_\ell v_\ell}{\partial t} + \frac{\partial \alpha_\ell\rho_\ell v_\ell^2}{\partial x} + \alpha_\ell\frac{\partial p}{\partial x} = \frac{v_{\mathrm{g}} + v_\ell}{2}\left(\frac{\partial \alpha_\ell\rho_\ell}{\partial t} + \frac{\partial \alpha_\ell\rho_\ell v_\ell}{\partial x}\right), \tag{3.4}$$

$$\frac{\partial(E_{\mathrm{g}} + E_\ell)}{\partial t} + \frac{\partial}{\partial x}((E_{\mathrm{g}} + \alpha_{\mathrm{g}}p)v_{\mathrm{g}} + (E_\ell + \alpha_\ell p)v_\ell) = 0. \tag{3.5}$$

Further, the internal energy equation becomes

$$\frac{\partial}{\partial t}\left(\alpha_{\mathrm{g}}\rho_{\mathrm{g}}e_{\mathrm{g}} + \alpha_\ell\rho_\ell e_\ell\right) + \frac{\partial}{\partial x}\left(\alpha_{\mathrm{g}}\rho_{\mathrm{g}}e_{\mathrm{g}}v_{\mathrm{g}} + \alpha_\ell\rho_\ell e_\ell v_\ell\right) + p\frac{\partial \alpha_{\mathrm{g}}v_{\mathrm{g}}}{\partial x} + p\frac{\partial \alpha_\ell v_\ell}{\partial x} = 0. \tag{3.6}$$

In the entropy equation (2.17), since $\mathscr{K}(\mu_{\mathrm{g}} - \mu_\ell)$ is finite, we have that $\mathscr{K}(\mu_{\mathrm{g}} - \mu_\ell)^2 \to 0$. The entropy equation becomes

$$\frac{\partial \alpha_{\mathrm{g}}\rho_{\mathrm{g}}s_{\mathrm{g}}}{\partial t} + \frac{\partial \alpha_\ell\rho_\ell s_\ell}{\partial t} + \frac{\partial \alpha_{\mathrm{g}}\rho_{\mathrm{g}}s_{\mathrm{g}}v_{\mathrm{g}}}{\partial x} + \frac{\partial \alpha_\ell\rho_\ell s_\ell v_\ell}{\partial x} = 0. \tag{3.7}$$

Now, to be able to have the model in quasilinear form, we first need to express the time derivatives $\partial_t\alpha_{\mathrm{g}}\rho_{\mathrm{g}}$ and $\partial_t\alpha_\ell\rho_\ell$ in terms of spatial derivatives.

3.1 Some differentials

Some useful differentials can be derived from the assumptions of equilibrium.

Proposition 2. *The differential of the pressure can be related to that of the temperature by*

$$\left(\frac{1}{\rho_{\mathrm{g}}} - \frac{1}{\rho_\ell}\right)\mathrm{d}p = \frac{L}{T}\,\mathrm{d}T, \tag{3.8}$$

where

$$L = e_{\mathrm{g}} + \frac{p}{\rho_{\mathrm{g}}} - e_\ell - \frac{p}{\rho_\ell} \tag{3.9}$$

is the latent heat.

Proof. From the expression of the thermodynamic potential (2.7) and the fundamental thermodynamic relation (2.14), we obtain

$$\mathrm{d}\mu = \frac{1}{\rho}\,\mathrm{d}p - s\,\mathrm{d}T. \tag{3.10}$$

Since $\mu_\mathrm{g} = \mu_\ell$, we can write

$$\left(\frac{1}{\rho_\mathrm{g}} - \frac{1}{\rho_\ell}\right)\mathrm{d}p = (s_\mathrm{g} - s_\ell)\,\mathrm{d}T. \tag{3.11}$$

Remark that, with the Clapeyron equation, we can write

$$s_\mathrm{g} - s_\ell = \frac{L}{T}. \tag{3.12}$$

Thus the differential becomes

$$\left(\frac{1}{\rho_\mathrm{g}} - \frac{1}{\rho_\ell}\right)\mathrm{d}p = \frac{L}{T}\,\mathrm{d}T. \tag{3.13}$$

□

Then, we can obtain simplified entropy and internal energy differentials.

Proposition 3. *The entropy differential for the gas phase is*

$$\mathrm{d}s_\mathrm{g} = -C_{p,\mathrm{g}}\chi_\mathrm{g}\,\mathrm{d}p, \tag{3.14}$$

and the internal energy differential for the liquid phase is

$$\mathrm{d}e_\mathrm{g} = \left(\frac{p}{\rho_\mathrm{g}^2 c_\mathrm{g}^2}\Psi_\mathrm{g} - TC_{p,\mathrm{g}}\chi_\mathrm{g}\right)\mathrm{d}p, \tag{3.15}$$

where

$$\chi_\mathrm{g} = \frac{\Gamma_\mathrm{g}}{\rho_\mathrm{g} c_\mathrm{g}^2} + \frac{\rho_\mathrm{g} - \rho_\ell}{\rho_\mathrm{g}\rho_\ell L}, \tag{3.16}$$

$$\chi_\ell = \frac{\Gamma_\ell}{\rho_\ell c_\ell^2} + \frac{\rho_\mathrm{g} - \rho_\ell}{\rho_\mathrm{g}\rho_\ell L}, \tag{3.17}$$

and

$$\Psi_\mathrm{g} = 1 + \rho_\mathrm{g} TC_{p,\mathrm{g}}\Gamma_\mathrm{g}\chi_\mathrm{g}, \tag{3.18}$$

$$\Psi_\ell = 1 + \rho_\ell TC_{p,\ell}\Gamma_\ell\chi_\ell. \tag{3.19}$$

The counterpart for the liquid phase of these differentials is found by symmetry of the phases.

Proof. An entropy differential may be found in (Flåtten & Lund 2011). For the gas phase, it reads

$$\mathrm{d}s_\mathrm{g} = -\frac{\Gamma_\mathrm{g} C_{p,\mathrm{g}}}{\rho_\mathrm{g} c_\mathrm{g}^2}\,\mathrm{d}p + \frac{C_{p,\mathrm{g}}}{T}\,\mathrm{d}T, \tag{3.20}$$

which with the help of (3.8) becomes

$$\mathrm{d}s_\mathrm{g} = -C_{p,\mathrm{g}}\left(\frac{\Gamma_\mathrm{g}}{\rho_\mathrm{g} c_\mathrm{g}^2} + \frac{\rho_\mathrm{g} - \rho_\ell}{\rho_\mathrm{g}\rho_\ell L}\right)\mathrm{d}p. \tag{3.21}$$

To simplify the results, the shorthands (3.16) and (3.18) have been defined for expressions which repetitively appear in the present article. This gives the result (3.14).

On the other hand, an internal energy differential may be found in (Flåtten *et al.* 2010). For the gas phase, it reads

$$\begin{aligned}\mathrm{d}e_\mathrm{g} &= \left(\frac{\partial e_\mathrm{g}}{\partial T}\right)_p \mathrm{d}T + \left(\frac{\partial e_\mathrm{g}}{\partial p}\right)_T \mathrm{d}p \\ &= C_{p,\mathrm{g}}\left(1 - \frac{\Gamma_\mathrm{g} p}{\rho_\mathrm{g} c_\mathrm{g}^2}\right)\mathrm{d}T + \left(\frac{p}{\rho_\mathrm{g}^2 c_\mathrm{g}^2} - \frac{\Gamma_\mathrm{g} T}{\rho_\mathrm{g} c_\mathrm{g}^2} C_{p,\mathrm{g}}\left(1 - \frac{\Gamma_\mathrm{g} p}{\rho_\mathrm{g} c_\mathrm{g}^2}\right)\right)\mathrm{d}p,\end{aligned} \tag{3.22}$$

which can be written through (3.8) as

$$\mathrm{d}e_{\mathrm{g}} = \frac{1}{\rho_{\mathrm{g}} c_{\mathrm{g}}^2}\left(\frac{p}{\rho_{\mathrm{g}}} - TC_{p,\mathrm{g}}(\rho_{\mathrm{g}} c_{\mathrm{g}}^2 - \Gamma_{\mathrm{g}} p)\left(\frac{\rho_{\mathrm{g}}-\rho_\ell}{\rho_{\mathrm{g}}\rho_\ell L} + \frac{\Gamma_{\mathrm{g}}}{\rho_{\mathrm{g}} c_{\mathrm{g}}^2}\right)\right)\mathrm{d}p. \tag{3.23}$$

Using the shorthands (3.16) and (3.18), this gives the result (3.15). Note that this expression may be written, through (3.14), as

$$\mathrm{d}e_{\mathrm{g}} = \frac{p}{\rho_{\mathrm{g}}^2 c_{\mathrm{g}}^2}\Psi_{\mathrm{g}}\,\mathrm{d}p + T\,\mathrm{d}s_{\mathrm{g}}. \tag{3.24}$$

□

3.2 Treatment of the time derivatives

The momentum equations (3.3) and (3.4) contain time derivatives, which we wish to convert to spatial derivatives.

Proposition 4. *The relaxed gas-phase mass equation may be written as*

$$\frac{\partial \alpha_{\mathrm{g}}\rho_{\mathrm{g}}}{\partial t} + \frac{\partial \alpha_{\mathrm{g}}\rho_{\mathrm{g}} v_{\mathrm{g}}}{\partial x} = -\mathscr{P}\frac{\partial p}{\partial x} - \mathscr{V}\left(\frac{\partial \alpha_{\mathrm{g}} v_{\mathrm{g}}}{\partial x} + \frac{\partial \alpha_\ell v_\ell}{\partial x}\right), \tag{3.25}$$

where

$$\mathscr{V} = \frac{T(\alpha_{\mathrm{g}}\rho_{\mathrm{g}} C_{p,\mathrm{g}}\chi_{\mathrm{g}} + \alpha_\ell\rho_\ell C_{p,\ell}\chi_\ell)}{L\left(\frac{\alpha_{\mathrm{g}}}{\rho_{\mathrm{g}} c_{\mathrm{g}}^2}\Psi_{\mathrm{g}} + \frac{\alpha_\ell}{\rho_\ell c_\ell^2}\Psi_\ell\right) + T(\alpha_{\mathrm{g}}\rho_{\mathrm{g}} C_{p,\mathrm{g}}\chi_{\mathrm{g}} + \alpha_\ell\rho_\ell C_{p,\ell}\chi_\ell)\frac{\rho_{\mathrm{g}}-\rho_\ell}{\rho_{\mathrm{g}}\rho_\ell}}, \tag{3.26}$$

$$\mathscr{P} = \frac{\alpha_{\mathrm{g}}\alpha_\ell T(v_{\mathrm{g}} - v_\ell)\left(\rho_\ell C_{p,\ell}\chi_\ell\frac{\Psi_{\mathrm{g}}}{\rho_{\mathrm{g}} c_{\mathrm{g}}^2} - \rho_{\mathrm{g}} C_{p,\mathrm{g}}\chi_{\mathrm{g}}\frac{\Psi_\ell}{\rho_\ell c_\ell^2}\right)}{L\left(\frac{\alpha_{\mathrm{g}}}{\rho_{\mathrm{g}} c_{\mathrm{g}}^2}\Psi_{\mathrm{g}} + \frac{\alpha_\ell}{\rho_\ell c_\ell^2}\Psi_\ell\right) + T(\alpha_{\mathrm{g}}\rho_{\mathrm{g}} C_{p,\mathrm{g}}\chi_{\mathrm{g}} + \alpha_\ell\rho_\ell C_{p,\ell}\chi_\ell)\frac{\rho_{\mathrm{g}}-\rho_\ell}{\rho_{\mathrm{g}}\rho_\ell}}. \tag{3.27}$$

This expression can be substituted in the momentum equation for the gas phase (3.3)*, thus eliminating the time derivatives. For the liquid phase, the relaxed mass equation reads*

$$\frac{\partial \alpha_\ell\rho_\ell}{\partial t} + \frac{\partial \alpha_\ell\rho_\ell v_\ell}{\partial x} = \mathscr{P}\frac{\partial p}{\partial x} + \mathscr{V}\left(\frac{\partial \alpha_{\mathrm{g}} v_{\mathrm{g}}}{\partial x} + \frac{\partial \alpha_\ell v_\ell}{\partial x}\right). \tag{3.28}$$

Proof. From the differentials (3.14) and (3.15) as well as the mixture mass equation (3.2), internal energy equation (3.6) and entropy equation (3.7), we are able to find three relations between $\partial_t p$, $\partial_t \alpha_{\mathrm{g}}\rho_{\mathrm{g}}$ and $\partial_t \alpha_\ell\rho_\ell$ and spatial derivatives. Therefore we can find an expression for each of the time derivatives.

The first relation is the mass equation (3.2)

$$\frac{\partial(\alpha_{\mathrm{g}}\rho_{\mathrm{g}} + \alpha_\ell\rho_\ell)}{\partial t} + \frac{\partial(\alpha_{\mathrm{g}}\rho_{\mathrm{g}} v_{\mathrm{g}} + \alpha_\ell\rho_\ell v_\ell)}{\partial x} = 0. \tag{3.29}$$

Then, the derivatives are expanded in the entropy equation (3.7). The derivatives $\partial_t s_{\mathrm{k}}$ and $\partial_x s_{\mathrm{k}}$ are subsequently substituted using the entropy differential (3.14) to obtain a second relation

$$\begin{aligned} -(\alpha_{\mathrm{g}}\rho_{\mathrm{g}} C_{p,\mathrm{g}}\chi_{\mathrm{g}} + \alpha_\ell\rho_\ell C_{p,\ell}\chi_\ell)\frac{\partial p}{\partial t} - (\alpha_{\mathrm{g}}\rho_{\mathrm{g}} C_{p,\mathrm{g}}\chi_{\mathrm{g}} v_{\mathrm{g}} + \alpha_\ell\rho_\ell C_{p,\ell}\chi_\ell v_\ell)\frac{\partial p}{\partial x} \\ + s_{\mathrm{g}}\frac{\partial \alpha_{\mathrm{g}}\rho_{\mathrm{g}}}{\partial t} + s_\ell\frac{\partial \alpha_\ell\rho_\ell}{\partial t} + s_{\mathrm{g}}\frac{\partial \alpha_{\mathrm{g}}\rho_{\mathrm{g}} v_{\mathrm{g}}}{\partial x} + s_\ell\frac{\partial \alpha_\ell\rho_\ell v_\ell}{\partial x} = 0. \end{aligned} \tag{3.30}$$

Finally, the same treatment is applied to the internal energy equation (3.6) with the differential (3.15), which gives a third relation

$$\begin{aligned} \left(\alpha_{\mathrm{g}}\rho_{\mathrm{g}}\left(\frac{p}{\rho_{\mathrm{g}}^2 c_{\mathrm{g}}^2}\Psi_{\mathrm{g}} - TC_{p,\mathrm{g}}\chi_{\mathrm{g}}\right) + \alpha_\ell\rho_\ell\left(\frac{p}{\rho_\ell^2 c_\ell^2}\Psi_\ell - TC_{p,\ell}\chi_\ell\right)\right)\frac{\partial p}{\partial t} \\ + \left(\alpha_{\mathrm{g}}\rho_{\mathrm{g}}\left(\frac{p}{\rho_{\mathrm{g}}^2 c_{\mathrm{g}}^2}\Psi_{\mathrm{g}} - TC_{p,\mathrm{g}}\chi_{\mathrm{g}}\right)v_{\mathrm{g}} + \alpha_\ell\rho_\ell\left(\frac{p}{\rho_\ell^2 c_\ell^2}\Psi_\ell - TC_{p,\ell}\chi_\ell\right)v_\ell\right)\frac{\partial p}{\partial x} \\ + e_{\mathrm{g}}\frac{\partial \alpha_{\mathrm{g}}\rho_{\mathrm{g}}}{\partial t} + e_\ell\frac{\partial \alpha_\ell\rho_\ell}{\partial t} + e_{\mathrm{g}}\frac{\partial \alpha_{\mathrm{g}}\rho_{\mathrm{g}} v_{\mathrm{g}}}{\partial x} + e_\ell\frac{\partial \alpha_\ell\rho_\ell v_\ell}{\partial x} + p\frac{\partial \alpha_{\mathrm{g}} v_{\mathrm{g}}}{\partial x} + p\frac{\partial \alpha_\ell v_\ell}{\partial x} = 0. \end{aligned} \tag{3.31}$$

Solving these three relations, we obtain the relaxed gas-phase mass equation (3.25). To find the equation for the liquid phase, we remark that the mixture mass equation (3.2) gives

$$\frac{\partial \alpha_\ell \rho_\ell}{\partial t} + \frac{\partial \alpha_\ell \rho_\ell v_\ell}{\partial x} = -\frac{\partial \alpha_g \rho_g}{\partial t} - \frac{\partial \alpha_g \rho_g v_g}{\partial x}, \tag{3.32}$$

which gives the result through (3.25). □

3.3 Regularising term

As with the six- and five-equation two-fluid models, we expect the present four-equation model not to be hyperbolic when the gas and liquid velocities are different from each other ((Gidaspow 1974, Stuhmiller 1977)). The eigenvalues associated with the volume-fraction waves are expected to be complex. We choose to include a regularising term similar to the interfacial-pressure regularising term for the six-equation two-fluid model ((Bestion 1990, Coquel *et al.* 1997, Cortes *et al.* 1998, Evje & Flåtten 2003, Paillère *et al.* 2003, Toumi 1996)). It consists in applying a pressure difference Δp between the two phases. The momentum equations are transformed into

$$\frac{\partial \alpha_g \rho_g v_g}{\partial t} + \frac{\partial \alpha_g \rho_g v_g^2}{\partial x} + \alpha_g \frac{\partial p}{\partial x} + \Delta p \frac{\partial \alpha_g}{\partial x} = \frac{v_g + v_\ell}{2} \left(\frac{\partial \alpha_g \rho_g}{\partial t} + \frac{\partial \alpha_g \rho_g v_g}{\partial x} \right), \tag{3.33}$$

and

$$\frac{\partial \alpha_\ell \rho_\ell v_\ell}{\partial t} + \frac{\partial \alpha_\ell \rho_\ell v_\ell^2}{\partial x} + \alpha_\ell \frac{\partial p}{\partial x} + \Delta p \frac{\partial \alpha_\ell}{\partial x} = \frac{v_g + v_\ell}{2} \left(\frac{\partial \alpha_\ell \rho_\ell}{\partial t} + \frac{\partial \alpha_\ell \rho_\ell v_\ell}{\partial x} \right), \tag{3.34}$$

while the mass and energy equations are not modified.

3.4 Expression of the model

As a result of the present section, the four-equation model (3.2)–(3.5) can be written, using (3.25), (3.28), (3.33) and (3.34), in the following form

$$\frac{\partial(\alpha_g \rho_g + \alpha_\ell \rho_\ell)}{\partial t} + \frac{\partial(\alpha_g \rho_g v_g + \alpha_\ell \rho_\ell v_\ell)}{\partial x} = 0, \tag{3.35}$$

$$\frac{\partial \alpha_g \rho_g v_g}{\partial t} + \frac{\partial \alpha_g \rho_g v_g^2}{\partial x} + \left(\alpha_g + \frac{v_g + v_\ell}{2} \mathscr{P} \right) \frac{\partial p}{\partial x} + \frac{v_g + v_\ell}{2} \mathscr{V} \frac{\partial(\alpha_g v_g + \alpha_\ell v_\ell)}{\partial x} + \Delta p \frac{\partial \alpha_g}{\partial x} = 0, \tag{3.36}$$

$$\frac{\partial \alpha_\ell \rho_\ell v_\ell}{\partial t} + \frac{\partial \alpha_\ell \rho_\ell v_\ell^2}{\partial x} + \left(\alpha_\ell - \frac{v_g + v_\ell}{2} \mathscr{P} \right) \frac{\partial p}{\partial x} - \frac{v_g + v_\ell}{2} \mathscr{V} \frac{\partial(\alpha_g v_g + \alpha_\ell v_\ell)}{\partial x} + \Delta p \frac{\partial \alpha_\ell}{\partial x} = 0, \tag{3.37}$$

$$\frac{\partial(E_g + E_\ell)}{\partial t} + \frac{\partial}{\partial x}((E_g + \alpha_g p) v_g + (E_\ell + \alpha_\ell p) v_\ell) = 0. \tag{3.38}$$

4 Quasilinear form

We wish to write the model in quasilinear form

$$\frac{\partial U}{\partial t} + A(U) \frac{\partial U}{\partial x} = 0, \tag{4.1}$$

where the vector of variables U is defined as

$$U = \begin{pmatrix} \alpha_g \rho_g + \alpha_\ell \rho_\ell \\ \alpha_g \rho_g v_g \\ \alpha_\ell \rho_\ell v_\ell \\ E_g + E_\ell \end{pmatrix}. \tag{4.2}$$

The matrix $A(U)$ is the Jacobian of the flux. The flux is split into a conservative and a non-conservative part, such that the system can be written as

$$\frac{\partial U}{\partial t} + \frac{\partial F_c(U)}{\partial x} + B(U) \frac{\partial W(U)}{\partial x} = 0, \tag{4.3}$$

where the conservative flux is

$$F_c(U) = \begin{pmatrix} \alpha_{\mathrm{g}}\rho_{\mathrm{g}}v_{\mathrm{g}} + \alpha_\ell\rho_\ell v_\ell \\ \alpha_{\mathrm{g}}\rho_{\mathrm{g}}v_{\mathrm{g}}^2 \\ \alpha_\ell\rho_\ell v_\ell^2 \\ (E_{\mathrm{g}} + \alpha_{\mathrm{g}}p)v_{\mathrm{g}} + (E_\ell + \alpha_\ell p)v_\ell \end{pmatrix}, \tag{4.4}$$

while the non-conservative contributions are

$$B(U) = \begin{pmatrix} 0 & 0 & 0 \\ \alpha_{\mathrm{g}} + \frac{v_{\mathrm{g}}+v_\ell}{2}\mathscr{P} & \frac{v_{\mathrm{g}}+v_\ell}{2}\mathscr{V} & \Delta p \\ \alpha_\ell - \frac{v_{\mathrm{g}}+v_\ell}{2}\mathscr{P} & -\frac{v_{\mathrm{g}}+v_\ell}{2}\mathscr{V} & -\Delta p \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} p \\ \alpha_{\mathrm{g}}v_{\mathrm{g}} + \alpha_\ell v_\ell \\ \alpha_{\mathrm{g}} \end{pmatrix}. \tag{4.5}$$

4.1 Some differentials

In order to write the Jacobian of the fluxes, we need to express the differentials of some variables in terms of the differential of the components of the variable vector U. We will find them with the help of the fundamental relation of thermodynamics (2.14) as well as the differentials of the components of the vector U. First, we will express all the differentials in terms of the differential of the gas density. Then, the other differentials will follow.

Proposition 5. *The density differential may be expressed in terms of the differentials of the variable-vector components u_i as*

$$\mathrm{d}\rho_{\mathrm{g}} = \frac{1}{\Phi}\frac{\Psi_{\mathrm{g}}}{c_{\mathrm{g}}^2}\left(\frac{\mathscr{E}}{\rho_{\mathrm{g}} - \rho_\ell}\mathrm{d}u_1 - v_{\mathrm{g}}\,\mathrm{d}u_2 - v_\ell\,\mathrm{d}u_3 + \mathrm{d}u_4\right) \tag{4.6}$$

where we have used the following shorthands

$$\Phi = \alpha_{\mathrm{g}}\frac{p}{\rho_{\mathrm{g}}c_{\mathrm{g}}^2}\Psi_{\mathrm{g}} + \alpha_\ell\frac{p}{\rho_\ell c_\ell^2}\Psi_\ell - \left(\alpha_{\mathrm{g}}\rho_{\mathrm{g}}TC_{p,\mathrm{g}}\chi_{\mathrm{g}} + \alpha_\ell\rho_\ell TC_{p,\ell}\chi_\ell\right)$$
$$+ \frac{1}{\rho_{\mathrm{g}} - \rho_\ell}\left(-e_{\mathrm{g}} + \frac{1}{2}v_{\mathrm{g}}^2 + e_\ell - \frac{1}{2}v_\ell^2\right)\left(\alpha_{\mathrm{g}}\rho_\ell\frac{\Psi_{\mathrm{g}}}{c_{\mathrm{g}}^2} + \alpha_\ell\rho_{\mathrm{g}}\frac{\Psi_\ell}{c_\ell^2}\right) \tag{4.7}$$

and

$$\mathscr{E} = -\rho_{\mathrm{g}}\left(e_{\mathrm{g}} - \frac{1}{2}v_{\mathrm{g}}^2\right) + \rho_\ell\left(e_\ell - \frac{1}{2}v_\ell^2\right). \tag{4.8}$$

Proof. We recall from the previous section the differential (3.24)

$$\mathrm{d}e_{\mathrm{g}} = \frac{p}{\rho_{\mathrm{g}}^2 c_{\mathrm{g}}^2}\Psi_{\mathrm{g}}\,\mathrm{d}p + T\,\mathrm{d}s_{\mathrm{g}}. \tag{4.9}$$

By identification with the fundamental thermodynamic relation (2.14), we can deduce

$$\Psi_{\mathrm{g}}\,\mathrm{d}p = c_{\mathrm{g}}^2\,\mathrm{d}\rho_{\mathrm{g}}, \tag{4.10}$$

and using the relation between pressure and temperature differentials (3.8), we obtain

$$-\Psi_{\mathrm{g}}\frac{\rho_{\mathrm{g}}\rho_\ell L}{T(\rho_{\mathrm{g}} - \rho_\ell)}\,\mathrm{d}T = c_{\mathrm{g}}^2\,\mathrm{d}\rho_{\mathrm{g}}. \tag{4.11}$$

Now, we write the differential of the thermodynamic potentials for both phases in terms of their respective density differentials, using (4.10) and (4.11)

$$\mathrm{d}\mu_{\mathrm{g}} = \frac{1}{\rho_{\mathrm{g}}}\mathrm{d}p - s_{\mathrm{g}}\,\mathrm{d}T = \frac{1}{\rho_{\mathrm{g}}}\frac{c_{\mathrm{g}}^2}{\Psi_{\mathrm{g}}}\mathrm{d}\rho_{\mathrm{g}} + s_{\mathrm{g}}\frac{c_{\mathrm{g}}^2}{\Psi_{\mathrm{g}}}\frac{T(\rho_{\mathrm{g}} - \rho_\ell)}{\rho_{\mathrm{g}}\rho_\ell L}\mathrm{d}\rho_{\mathrm{g}} \tag{4.12}$$

$$\mathrm{d}\mu_\ell = \frac{1}{\rho_\ell}\mathrm{d}p - s_\ell\,\mathrm{d}T = \frac{1}{\rho_\ell}\frac{c_\ell^2}{\Psi_\ell}\mathrm{d}\rho_\ell + s_\ell\frac{c_\ell^2}{\Psi_\ell}\frac{T(\rho_{\mathrm{g}} - \rho_\ell)}{\rho_{\mathrm{g}}\rho_\ell L}\mathrm{d}\rho_\ell \tag{4.13}$$

and equate them, using the assumption of chemical equilibrium. Implicitly, we also use the mechanical and thermal equilibrium assumptions, since we have expressed the pressure and temperature differentials in terms of the gas as well as of the liquid phase variables. This gives a relation between the density differentials:

$$\frac{c_{\mathrm{g}}^2}{\Psi_{\mathrm{g}}}\mathrm{d}\rho_{\mathrm{g}} = \frac{c_\ell^2}{\Psi_\ell}\mathrm{d}\rho_\ell. \tag{4.14}$$

Next, we need a relation for the energy differentials. For the gas phase, we find it using the differential of $p(\rho_\mathrm{g}, e_\mathrm{g})$

$$\mathrm{d}p = \left(c_\mathrm{g}^2 - \Gamma_\mathrm{g}\frac{p}{\rho_\mathrm{g}}\right)\mathrm{d}\rho_\mathrm{g} + \Gamma_\mathrm{g}\rho_\mathrm{g}\,\mathrm{d}e_\mathrm{g}, \tag{4.15}$$

where $\mathrm{d}p$ is replaced using (3.15). After simplification, we obtain

$$\frac{\Psi_\mathrm{g}}{\left(\frac{p}{\rho_\mathrm{g}^2 c_\mathrm{g}^2}\Psi_\mathrm{g} - TC_{p,\mathrm{g}}\chi_\mathrm{g}\right)}\mathrm{d}e_\mathrm{g} = c_\mathrm{g}^2\,\mathrm{d}\rho_\mathrm{g}. \tag{4.16}$$

For the liquid phase, we first use the phase symmetry to obtain

$$\frac{\Psi_\ell}{\left(\frac{p}{\rho_\ell^2 c_\ell^2}\Psi_\ell - TC_{p,\ell}\chi_\ell\right)}\mathrm{d}e_\ell = c_\ell^2\,\mathrm{d}\rho_\ell, \tag{4.17}$$

and then replace the liquid density differential using (4.14)

$$\frac{1}{\left(\frac{p}{\rho_\ell^2 c_\ell^2}\Psi_\ell - TC_{p,\ell}\chi_\ell\right)}\mathrm{d}e_\ell = \frac{c_\mathrm{g}^2}{\Psi_\mathrm{g}}\,\mathrm{d}\rho_\mathrm{g}. \tag{4.18}$$

Further, we seek an expression for the differential of the volume fraction. From the differential of the first component of the vector U, we have

$$\mathrm{d}u_1 = \alpha_\mathrm{g}\,\mathrm{d}\rho_\mathrm{g} + \alpha_\ell\,\mathrm{d}\rho_\ell + (\rho_\mathrm{g} - \rho_\ell)\,\mathrm{d}\alpha_\mathrm{g}, \tag{4.19}$$

where ρ_ℓ is eliminated using the differential (4.14)

$$(\rho_\mathrm{g} - \rho_\ell)\,\mathrm{d}\alpha_\mathrm{g} = \mathrm{d}u_1 - \left(\alpha_\mathrm{g} + \alpha_\ell\frac{c_\mathrm{g}^2\Psi_\ell}{c_\ell^2\Psi_\mathrm{g}}\right)\mathrm{d}\rho_\mathrm{g}. \tag{4.20}$$

Finally, we would like to find an expression for the velocity differentials. For the gas phase, we start from the differential of the second component of the vector U

$$\mathrm{d}u_2 = \mathrm{d}(\alpha_\mathrm{g}\rho_\mathrm{g}v_\mathrm{g}) = \alpha_\mathrm{g}\rho_\mathrm{g}\,\mathrm{d}v_\mathrm{g} + \alpha_\mathrm{g}v_\mathrm{g}\,\mathrm{d}\rho_\mathrm{g} + \rho_\mathrm{g}v_\mathrm{g}\,\mathrm{d}\alpha_\mathrm{g}, \tag{4.21}$$

where $\mathrm{d}\alpha_\mathrm{g}$ is replaced using (4.20) to obtain

$$\alpha_\mathrm{g}\rho_\mathrm{g}\,\mathrm{d}v_\mathrm{g} = -\frac{\rho_\mathrm{g}v_\mathrm{g}}{\rho_\mathrm{g} - \rho_\ell}\mathrm{d}u_1 + \mathrm{d}u_2 + \frac{v_\mathrm{g}}{\rho_\mathrm{g} - \rho_\ell}\left(\alpha_\mathrm{g}\rho_\ell + \alpha_\ell\rho_\mathrm{g}\frac{c_\mathrm{g}^2\Psi_\ell}{c_\ell^2\Psi_\mathrm{g}}\right)\mathrm{d}\rho_\mathrm{g}. \tag{4.22}$$

By phase symmetry, we deduce that

$$\alpha_\ell\rho_\ell\,\mathrm{d}v_\ell = \frac{\rho_\ell v_\ell}{\rho_\mathrm{g} - \rho_\ell}\mathrm{d}u_1 + \mathrm{d}u_3 - \frac{v_\ell}{\rho_\mathrm{g} - \rho_\ell}\left(\alpha_\ell\rho_\mathrm{g} + \alpha_\mathrm{g}\rho_\ell\frac{c_\ell^2\Psi_\mathrm{g}}{c_\mathrm{g}^2\Psi_\ell}\right)\mathrm{d}\rho_\ell. \tag{4.23}$$

In order to express it in terms of the differential for the gas density, we use (4.14) to obtain

$$\alpha_\ell\rho_\ell\,\mathrm{d}v_\ell = \frac{\rho_\ell v_\ell}{\rho_\mathrm{g} - \rho_\ell}\mathrm{d}u_1 + \mathrm{d}u_3 - \frac{v_\ell}{\rho_\mathrm{g} - \rho_\ell}\left(\alpha_\mathrm{g}\rho_\ell + \alpha_\ell\rho_\mathrm{g}\frac{c_\mathrm{g}^2\Psi_\ell}{c_\ell^2\Psi_\mathrm{g}}\right)\mathrm{d}\rho_\mathrm{g}. \tag{4.24}$$

Now, using the differential of the mixture internal energy, we are able to deduce a differential for the gas density $\mathrm{d}\rho_\mathrm{g}$. We have that

$$\mathrm{d}(\alpha_\mathrm{g}\rho_\mathrm{g}e_\mathrm{g}) + \mathrm{d}(\alpha_\ell\rho_\ell e_\ell) = \mathrm{d}u_4 - \frac{v_\mathrm{g}}{2}\mathrm{d}u_2 - \frac{v_\ell}{2}\mathrm{d}u_3 - \frac{1}{2}\alpha_\mathrm{g}\rho_\mathrm{g}v_\mathrm{g}\,\mathrm{d}v_\mathrm{g} - \frac{1}{2}\alpha_\ell\rho_\ell v_\ell\,\mathrm{d}v_\ell. \tag{4.25}$$

After having replaced all the differentials using the expressions (4.14), (4.16), (4.18), (4.20), (4.22) and (4.24) previously derived, we obtain the density differential 4.6. □

All the other differentials now follow. The differential of the volume fraction follows from (4.20) in which $\mathrm{d}\rho_\mathrm{g}$ is replaced using (4.6)

$$\mathrm{d}\alpha_\mathrm{g} = \frac{1}{\rho_\mathrm{g}-\rho_\ell}\,\mathrm{d}u_1 - \frac{1}{\rho_\mathrm{g}-\rho_\ell}\frac{1}{\Phi}\left(\alpha_\mathrm{g}\frac{\Psi_\mathrm{g}}{c_\mathrm{g}^2}+\alpha_\ell\frac{\Psi_\ell}{c_\ell^2}\right)\left(\frac{\mathscr{E}}{\rho_\mathrm{g}-\rho_\ell}\,\mathrm{d}u_1 - v_\mathrm{g}\,\mathrm{d}u_2 - v_\ell\,\mathrm{d}u_3 + \mathrm{d}u_4\right). \tag{4.26}$$

The differential of the pressure follows from (4.10)

$$\mathrm{d}p = \frac{1}{\Phi}\left(\frac{\mathscr{E}}{\rho_\mathrm{g}-\rho_\ell}\,\mathrm{d}u_1 - v_\mathrm{g}\,\mathrm{d}u_2 - v_\ell\,\mathrm{d}u_3 + \mathrm{d}u_4\right). \tag{4.27}$$

The differential of the liquid density follows from (4.14)

$$\mathrm{d}\rho_\ell = \frac{1}{\Phi}\frac{\Psi_\ell}{c_\ell^2}\left(\frac{\mathscr{E}}{\rho_\mathrm{g}-\rho_\ell}\,\mathrm{d}u_1 - v_\mathrm{g}\,\mathrm{d}u_2 - v_\ell\,\mathrm{d}u_3 + \mathrm{d}u_4\right). \tag{4.28}$$

The differentials of the internal energies follow from (4.16) and (4.18)

$$\mathrm{d}e_\mathrm{g} = \frac{1}{\Phi}\left(\frac{p}{\rho_\mathrm{g}^2 c_\mathrm{g}^2}\Psi_\mathrm{g} - TC_{p,\mathrm{g}}\chi_\mathrm{g}\right)\left(\frac{\mathscr{E}}{\rho_\mathrm{g}-\rho_\ell}\,\mathrm{d}u_1 - v_\mathrm{g}\,\mathrm{d}u_2 - v_\ell\,\mathrm{d}u_3 + \mathrm{d}u_4\right), \tag{4.29}$$

$$\mathrm{d}e_\ell = \frac{1}{\Phi}\left(\frac{p}{\rho_\ell^2 c_\ell^2}\Psi_\ell - TC_{p,\ell}\chi_\ell\right)\left(\frac{\mathscr{E}}{\rho_\mathrm{g}-\rho_\ell}\,\mathrm{d}u_1 - v_\mathrm{g}\,\mathrm{d}u_2 - v_\ell\,\mathrm{d}u_3 + \mathrm{d}u_4\right). \tag{4.30}$$

The differentials of the velocities follow from (4.22) and (4.24)

$$\begin{aligned}\alpha_\mathrm{g}\rho_\mathrm{g}\,\mathrm{d}v_\mathrm{g} = &-\frac{\rho_\mathrm{g}v_\mathrm{g}}{\rho_\mathrm{g}-\rho_\ell}\,\mathrm{d}u_1 + \mathrm{d}u_2 + \frac{1}{\Phi}\frac{v_\mathrm{g}}{\rho_\mathrm{g}-\rho_\ell}\\ &\cdot\left(\alpha_\mathrm{g}\rho_\ell\frac{\Psi_\mathrm{g}}{c_\mathrm{g}^2}+\alpha_\ell\rho_\mathrm{g}\frac{\Psi_\ell}{c_\ell^2}\right)\left(\frac{\mathscr{E}}{\rho_\mathrm{g}-\rho_\ell}\,\mathrm{d}u_1 - v_\mathrm{g}\,\mathrm{d}u_2 - v_\ell\,\mathrm{d}u_3 + \mathrm{d}u_4\right),\end{aligned} \tag{4.31}$$

$$\begin{aligned}\alpha_\ell\rho_\ell\,\mathrm{d}v_\ell = &\frac{\rho_\ell v_\ell}{\rho_\mathrm{g}-\rho_\ell}\,\mathrm{d}u_1 + \mathrm{d}u_3 - \frac{1}{\Phi}\frac{v_\ell}{\rho_\mathrm{g}-\rho_\ell}\\ &\cdot\left(\alpha_\mathrm{g}\rho_\ell\frac{\Psi_\mathrm{g}}{c_\mathrm{g}^2}+\alpha_\ell\rho_\mathrm{g}\frac{\Psi_\ell}{c_\ell^2}\right)\left(\frac{\mathscr{E}}{\rho_\mathrm{g}-\rho_\ell}\,\mathrm{d}u_1 - v_\mathrm{g}\,\mathrm{d}u_2 - v_\ell\,\mathrm{d}u_3 + \mathrm{d}u_4\right).\end{aligned} \tag{4.32}$$

4.2 Jacobian of the fluxes

We are now able to derive the Jacobian of the conservative fluxes $F_c(U)$ (4.4) and of the vector $W(U)$ in the non-conservative fluxes (4.5). To do so, we express the differentials of the components of the vectors $F_c(U)$ and $W(U)$ in terms of the differentials of the components of U. First, we simply have

$$\mathrm{d}f_1 = \mathrm{d}(\alpha_\mathrm{g}\rho_\mathrm{g}v_\mathrm{g}+\alpha_\ell\rho_\ell v_\ell) = \mathrm{d}u_2 + \mathrm{d}u_3. \tag{4.33}$$

Then for the second component

$$\mathrm{d}f_2 = \mathrm{d}(\alpha_\mathrm{g}\rho_\mathrm{g}v_\mathrm{g}^2) = v_\mathrm{g}\,\mathrm{d}(\alpha_\mathrm{g}\rho_\mathrm{g}v_\mathrm{g}) + \alpha_\mathrm{g}\rho_\mathrm{g}v_\mathrm{g}\,\mathrm{d}v_\mathrm{g} = v_\mathrm{g}\,\mathrm{d}u_2 + \alpha_\mathrm{g}\rho_\mathrm{g}v_\mathrm{g}\,\mathrm{d}v_\mathrm{g}, \tag{4.34}$$

where $\mathrm{d}v_\mathrm{g}$ is substituted using (4.31)

$$\begin{aligned}\mathrm{d}f_2 = &-\frac{\rho_\mathrm{g}v_\mathrm{g}^2}{\rho_\mathrm{g}-\rho_\ell}\,\mathrm{d}u_1 + 2v_\mathrm{g}\,\mathrm{d}u_2 + \frac{1}{\Phi}\frac{v_\mathrm{g}^2}{\rho_\mathrm{g}-\rho_\ell}\\ &\cdot\left(\alpha_\mathrm{g}\rho_\ell\frac{\Psi_\mathrm{g}}{c_\mathrm{g}^2}+\alpha_\ell\rho_\mathrm{g}\frac{\Psi_\ell}{c_\ell^2}\right)\left(\frac{\mathscr{E}}{\rho_\mathrm{g}-\rho_\ell}\,\mathrm{d}u_1 - v_\mathrm{g}\,\mathrm{d}u_2 - v_\ell\,\mathrm{d}u_3 + \mathrm{d}u_4\right).\end{aligned} \tag{4.35}$$

Similarly, for the third component

$$\mathrm{d}f_3 = v_\ell\,\mathrm{d}u_3 + \alpha_\ell\rho_\ell v_\ell\,\mathrm{d}v_\ell, \tag{4.36}$$

where $\mathrm{d}v_\ell$ is substituted using (4.32)

$$\mathrm{d}f_3 = \frac{\rho_\ell v_\ell^2}{\rho_\mathrm{g}-\rho_\ell}\,\mathrm{d}u_1 + 2v_\ell\,\mathrm{d}u_3 - \frac{1}{\Phi}\frac{v_\ell^2}{\rho_\mathrm{g}-\rho_\ell} \cdot \left(\alpha_\mathrm{g}\rho_\ell\frac{\Psi_\mathrm{g}}{c_\mathrm{g}^2} + \alpha_\ell\rho_\mathrm{g}\frac{\Psi_\ell}{c_\ell^2}\right)\left(\frac{\mathscr{E}}{\rho_\mathrm{g}-\rho_\ell}\,\mathrm{d}u_1 - v_\mathrm{g}\,\mathrm{d}u_2 - v_\ell\,\mathrm{d}u_3 + \mathrm{d}u_4\right). \tag{4.37}$$

Finally, the fourth component can be written as

$$\begin{aligned}\mathrm{d}f_4 = {}&\frac{1}{2}v_\mathrm{g}^2\,\mathrm{d}u_2 + \frac{1}{2}v_\ell^2\,\mathrm{d}u_3 + (\rho_\mathrm{g}e_\mathrm{g}v_\mathrm{g} + v_\mathrm{g}p - \rho_\ell e_\ell v_\ell - v_\ell p)\,\mathrm{d}\alpha_\mathrm{g}\\ &+ (\alpha_\mathrm{g}v_\mathrm{g} + \alpha_\ell v_\ell)\,\mathrm{d}p + \alpha_\mathrm{g}e_\mathrm{g}v_\mathrm{g}\,\mathrm{d}\rho_\mathrm{g} + \alpha_\ell e_\ell v_\ell\,\mathrm{d}\rho_\ell + \alpha_\mathrm{g}\rho_\mathrm{g}v_\mathrm{g}\,\mathrm{d}e_\mathrm{g} + \alpha_\ell\rho_\ell v_\ell\,\mathrm{d}e_\ell\\ &+ \left(\alpha_\mathrm{g}\rho_\mathrm{g}\left(e_\mathrm{g}+v_\mathrm{g}^2\right) + \alpha_\mathrm{g}p\right)\mathrm{d}v_\mathrm{g} + \left(\alpha_\ell\rho_\ell\left(e_\ell+v_\ell^2\right) + \alpha_\ell p\right)\mathrm{d}v_\ell,\end{aligned} \tag{4.38}$$

which after replacement of the differentials and simplification becomes

$$\begin{aligned}\mathrm{d}f_4 = {}&\frac{-\rho_\mathrm{g}v_\mathrm{g}^3 + \rho_\ell v_\ell^3}{\rho_\mathrm{g}-\rho_\ell}\,\mathrm{d}u_1 + \left(e_\mathrm{g} + \frac{3}{2}v_\mathrm{g}^2 + \frac{p}{\rho_\mathrm{g}}\right)\mathrm{d}u_2 + \left(e_\ell + \frac{3}{2}v_\ell^2 + \frac{p}{\rho_\ell}\right)\mathrm{d}u_3\\ &+ \frac{1}{\Phi}\left[\frac{v_\mathrm{g}^3 - v_\ell^3}{\rho_\mathrm{g}-\rho_\ell}\left(\alpha_\mathrm{g}\rho_\ell\frac{\Psi_\mathrm{g}}{c_\mathrm{g}^2} + \alpha_\ell\rho_\mathrm{g}\frac{\Psi_\ell}{c_\ell^2}\right) + \alpha_\mathrm{g}v_\mathrm{g} + \alpha_\ell v_\ell - T(\alpha_\mathrm{g}\rho_\mathrm{g}v_\mathrm{g}C_{p,\mathrm{g}}\chi_\mathrm{g} + \alpha_\ell\rho_\ell v_\ell C_{p,\ell}\chi_\ell)\right]\\ &\cdot\left(\frac{\mathscr{E}}{\rho_\mathrm{g}-\rho_\ell}\,\mathrm{d}u_1 - v_\mathrm{g}\,\mathrm{d}u_2 - v_\ell\,\mathrm{d}u_3 + \mathrm{d}u_4\right).\end{aligned} \tag{4.39}$$

Similarly, for the non-conservative part of the fluxes, we need to derive a Jacobian matrix for the vector W. First, we can remark that

$$\mathrm{d}w_1 = \mathrm{d}p, \tag{4.40}$$

which gives after substitution of the differentials

$$\mathrm{d}w_1 = \frac{1}{\Phi}\left(\frac{\mathscr{E}}{\rho_\mathrm{g}-\rho_\ell}\,\mathrm{d}u_1 - v_\mathrm{g}\,\mathrm{d}u_2 - v_\ell\,\mathrm{d}u_3 + \mathrm{d}u_4\right). \tag{4.41}$$

For the second component, we have that

$$\mathrm{d}w_2 = \mathrm{d}(\alpha_\mathrm{g}v_\mathrm{g} + \alpha_\ell v_\ell) = \frac{1}{\rho_\mathrm{g}}(\mathrm{d}u_2 - \alpha_\mathrm{g}v_\mathrm{g}\,\mathrm{d}\rho_\mathrm{g}) + \frac{1}{\rho_\ell}(\mathrm{d}u_3 - \alpha_\ell v_\ell\,\mathrm{d}\rho_\ell), \tag{4.42}$$

which gives

$$\mathrm{d}w_2 = \frac{1}{\rho_\mathrm{g}}\,\mathrm{d}u_2 + \frac{1}{\rho_\ell}\,\mathrm{d}u_3 - \frac{1}{\Phi}\left(\alpha_\mathrm{g}v_\mathrm{g}\frac{\Psi_\mathrm{g}}{\rho_\mathrm{g}c_\mathrm{g}^2} + \alpha_\ell v_\ell\frac{\Psi_\ell}{\rho_\ell c_\ell^2}\right) \cdot\left(\frac{\mathscr{E}}{\rho_\mathrm{g}-\rho_\ell}\,\mathrm{d}u_1 - v_\mathrm{g}\,\mathrm{d}u_2 - v_\ell\,\mathrm{d}u_3 + \mathrm{d}u_4\right). \tag{4.43}$$

Finally, the third component is the volume fraction differential (4.26)

$$\mathrm{d}w_3 = \mathrm{d}\alpha_\mathrm{g}, \tag{4.44}$$

thus

$$\mathrm{d}w_3 = \frac{1}{\rho_\mathrm{g}-\rho_\ell}\,\mathrm{d}u_1 - \frac{1}{\rho_\mathrm{g}-\rho_\ell}\frac{1}{\Phi}\left(\alpha_\mathrm{g}\frac{\Psi_\mathrm{g}}{c_\mathrm{g}^2} + \alpha_\ell\frac{\Psi_\ell}{c_\ell^2}\right)\cdot\left(\frac{\mathscr{E}}{\rho_\mathrm{g}-\rho_\ell}\,\mathrm{d}u_1 - v_\mathrm{g}\,\mathrm{d}u_2 - v_\ell\,\mathrm{d}u_3 + \mathrm{d}u_4\right). \tag{4.45}$$

4.3 The matrices in the quasilinear form

We can now write the matrix $A(U)$ appearing in the quasilinear form (4.1). Following a flux-splitting strategy (see for example (Evje & Flåtten 2003)), we may split the matrix in a conservative part and a non-conservative part.

With the help of (4.33), (4.35), (4.37) and (4.39), the conservative part is written as

$$A_c(U) = \frac{\partial F_c(U)}{\partial U} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -\frac{\rho_g v_g^2}{\rho_g-\rho_\ell} + \frac{v_g^2 \mathscr{E}}{\rho_g-\rho_\ell}\Sigma_\rho & 2v_g - v_g^3\Sigma_\rho & -v_g^2 v_\ell \Sigma_\rho & v_g^2\Sigma_\rho \\ \frac{\rho_\ell v_\ell^2}{\rho_g-\rho_\ell} - \frac{v_\ell^2 \mathscr{E}}{\rho_g-\rho_\ell}\Sigma_\rho & v_g v_\ell^2 \Sigma_\rho & 2v_\ell + v_\ell^3\Sigma_\rho & -v_\ell^2\Sigma_\rho \\ a_{41} & a_{42} & a_{43} & (v_g^3 - v_\ell^3)\Sigma_\rho + \Omega \end{pmatrix} \tag{4.46}$$

where

$$a_{41} = \frac{-\rho_g v_g^3 + \rho_\ell v_\ell^3}{\rho_g - \rho_\ell} + \left((v_g^3 - v_\ell^3)\Sigma_\rho + \Omega\right)\frac{\mathscr{E}}{\rho_g - \rho_\ell}, \tag{4.47}$$

$$a_{42} = \left(e_g + \frac{3}{2}v_g^2 + \frac{p}{\rho_g}\right) - \left((v_g^3 - v_\ell^3)\Sigma_\rho + \Omega\right) v_g, \tag{4.48}$$

$$a_{43} = \left(e_\ell + \frac{3}{2}v_\ell^2 + \frac{p}{\rho_\ell}\right) - \left((v_g^3 - v_\ell^3)\Sigma_\rho + \Omega\right) v_\ell. \tag{4.49}$$

We have also introduced the shorthands

$$\Sigma_\rho = \frac{1}{\Phi}\frac{1}{(\rho_g - \rho_\ell)}\left(\alpha_g \rho_\ell \frac{\Psi_g}{c_g^2} + \alpha_\ell \rho_g \frac{\Psi_\ell}{c_\ell^2}\right) \tag{4.50}$$

and

$$\Omega = \frac{1}{\Phi}\left(\alpha_g v_g + \alpha_\ell v_\ell - \alpha_g \rho_g v_g T C_{p,g}\chi_g - \alpha_\ell \rho_\ell v_\ell T C_{p,\ell}\chi_\ell\right). \tag{4.51}$$

For the non-conservative part, we can express the Jacobian of the vector $W(U)$ using (4.41), (4.43) and (4.45)

$$M(U) = \frac{\partial W(U)}{\partial U} = \begin{pmatrix} \frac{1}{\Phi}\frac{\mathscr{E}}{\rho_g-\rho_\ell} & -\frac{v_g}{\Phi} & -\frac{v_\ell}{\Phi} & \frac{1}{\Phi} \\ -\frac{\mathscr{E}}{\rho_g-\rho_\ell}\Sigma_v & \frac{1}{\rho_g} + v_g\Sigma_v & \frac{1}{\rho_\ell} + v_\ell\Sigma_v & -\Sigma_v \\ \frac{1}{\rho_g-\rho_\ell} - \frac{\mathscr{E}}{\rho_g-\rho_\ell}\Sigma & v_g\Sigma & v_\ell\Sigma & -\Sigma \end{pmatrix} \tag{4.52}$$

where

$$\Sigma = \frac{1}{\Phi}\frac{1}{\rho_g - \rho_\ell}\left(\alpha_g \frac{\Psi_g}{c_g^2} + \alpha_\ell \frac{\Psi_\ell}{c_\ell^2}\right), \tag{4.53}$$

$$\Sigma_v = \frac{1}{\Phi}\left(\alpha_g v_g \frac{\Psi_g}{\rho_g c_g^2} + \alpha_\ell v_\ell \frac{\Psi_\ell}{\rho_\ell c_\ell^2}\right). \tag{4.54}$$

The Jacobian of the non-conservative fluxes then follows from

$$A_p(U) = B(U)\cdot M(U). \tag{4.55}$$

The Jacobian of the whole system is then

$$A(U) = A_c(U) + A_p(U). \tag{4.56}$$

5 Subcharacteristic condition

The subcharacteristic condition is a stability condition which states that the stiff limit of a relaxation model – called the equilibrium model – can only be stable if the wave speeds of the equilibrium system do not exceed the speeds of the corresponding waves of its relaxation system ((Chen *et al.* 1994, Flåtten & Lund 2011, Liu 1987a, Natalini 1999)). We expect the two-fluid models mentioned in the present paper to respect this condition since the underlying physical models describe a stable reality. Figure 1 presents the model hierarchy, where TF and DF, respectively, denote the two-fluid and the drift-flux models, and the index, the number of conservation equations in the model. Each arrow designates the relaxation performed from one model to the next. The subcharacteristic condition has been proved for some of the relaxation processes by (Martínez Ferrer *et al.* 2012) and (Flåtten & Lund 2011). In the present section, we prove the subcharacteristic condition for the remaining relaxation processes $TF_5 \to TF_4$ and $TF_4 \to DF_3$.

$$
\begin{tikzcd}
\mathrm{TF}_7 \arrow[r, "p"] & \mathrm{TF}_6 \arrow[r, "T"] \arrow[d, "v"] & \mathrm{TF}_5 \arrow[r, "\mu"] \arrow[d, "v"] & \mathrm{TF}_4 \arrow[d, "v"] \\
 & \mathrm{DF}_5 \arrow[r, "T"] & \mathrm{DF}_4 \arrow[r, "\mu"] & \mathrm{DF}_3
\end{tikzcd}
$$

Figure 1: Hierarchy of the two-phase flow models. TF: two-phase model. DF: drift-flux model. Index: Number of conservation equations.

5.1 Speed of sound

The eigenvalues of the Jacobian of the fluxes $A(U)$ are the propagation velocities of the quantities defined by the eigenvectors of $A(U)$, also called waves. In the present model, these waves are the volume-fraction waves and the pressure waves.

Proposition 6. *When the liquid and gas velocities are equal to each other, the eigenvalues of the two-fluid four-equation model are*

$$\Lambda_{TF4} = \begin{pmatrix} v_{\mathrm{m}} - c_{TF4} \\ v_{\mathrm{m}} \\ v_{\mathrm{m}} \\ v_{\mathrm{m}} + c_{TF4} \end{pmatrix}, \tag{5.1}$$

where the velocities have been substituted by $v_{\mathrm{g}} = v_{\mathrm{m}}$ and $v_\ell = v_{\mathrm{m}}$, and the speed of sound of the model is given by

$$c_{TF4} = \sqrt{\frac{\alpha_\ell\rho_{\mathrm{g}} + \alpha_{\mathrm{g}}\rho_\ell}{\rho_{\mathrm{g}}\rho_\ell\left(\frac{\alpha_{\mathrm{g}}}{\rho_{\mathrm{g}}c_{\mathrm{g}}^2} + \frac{\alpha_\ell}{\rho_\ell c_\ell^2} + T\left(\alpha_{\mathrm{g}}\rho_{\mathrm{g}}C_{p,\mathrm{g}}\chi_{\mathrm{g}}^2 + \alpha_\ell\rho_\ell C_{p,\ell}\chi_\ell^2\right)\right)}}. \tag{5.2}$$

Proof. When $v_{\mathrm{g}} = 0$ and $v_\ell = 0$, the matrix $A(U)$ becomes

$$A(U(v_{\mathrm{g}} = 0, v_\ell = 0)) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \frac{\alpha_{\mathrm{g}}(\rho_{\mathrm{g}}e_{\mathrm{g}} - \rho_\ell e_\ell)c_{\mathrm{TF4}}^2}{(\alpha_\ell\rho_{\mathrm{g}} + \alpha_{\mathrm{g}}\rho_\ell)T(s_{\mathrm{g}} - s_\ell)} & 0 & 0 & -\frac{\alpha_{\mathrm{g}}(\rho_{\mathrm{g}} - \rho_\ell)c_{\mathrm{TF4}}^2}{(\alpha_\ell\rho_{\mathrm{g}} + \alpha_{\mathrm{g}}\rho_\ell)T(s_{\mathrm{g}} - s_\ell)} \\ \frac{\alpha_\ell(\rho_{\mathrm{g}}e_{\mathrm{g}} - \rho_\ell e_\ell)c_{\mathrm{TF4}}^2}{(\alpha_\ell\rho_{\mathrm{g}} + \alpha_{\mathrm{g}}\rho_\ell)T(s_{\mathrm{g}} - s_\ell)} & 0 & 0 & -\frac{\alpha_\ell(\rho_{\mathrm{g}} - \rho_\ell)c_{\mathrm{TF4}}^2}{(\alpha_\ell\rho_{\mathrm{g}} + \alpha_{\mathrm{g}}\rho_\ell)T(s_{\mathrm{g}} - s_\ell)} \\ 0 & e_{\mathrm{g}} + \frac{p}{\rho_{\mathrm{g}}} & e_\ell + \frac{p}{\rho_\ell} & 0 \end{pmatrix}, \tag{5.3}$$

where

$$c_{\mathrm{TF4}} = \sqrt{\frac{\alpha_\ell\rho_{\mathrm{g}} + \alpha_{\mathrm{g}}\rho_\ell}{\rho_{\mathrm{g}}\rho_\ell\left(\frac{\alpha_{\mathrm{g}}}{\rho_{\mathrm{g}}c_{\mathrm{g}}^2}\Psi_{\mathrm{g}} + \frac{\alpha_\ell}{\rho_\ell c_\ell^2}\Psi_\ell + T\left(\alpha_{\mathrm{g}}\rho_{\mathrm{g}}C_{p,\mathrm{g}}\chi_{\mathrm{g}} + \alpha_\ell\rho_\ell C_{p,\ell}\chi_\ell\right)\frac{\rho_{\mathrm{g}} - \rho_\ell}{\rho_{\mathrm{g}}\rho_\ell L}\right)}}. \tag{5.4}$$

Its eigenvalues are then $0, 0, c_{\mathrm{TF4}}$ and $-c_{\mathrm{TF4}}$. The waves with zero velocity are the volume-fraction waves, while the two other are the pressure waves. We deduce that c_{TF4} is the speed of sound of the model. This speed of sound is dependent on the thermodynamical assumptions, here that the phases are at all times at equilibrium. The expression (5.4) uses the variable blocks that are involved in the Jacobian matrices. We can also reorganise it to the more compact form 5.2.

Note that the speed of sound can be used to simplify (3.26) and (3.27)

$$\mathscr{V} = \frac{\rho_{\mathrm{g}}\rho_\ell}{\alpha_\ell\rho_{\mathrm{g}} + \alpha_{\mathrm{g}}\rho_\ell}\frac{T}{L}(\alpha_{\mathrm{g}}\rho_{\mathrm{g}}C_{p,\mathrm{g}}\chi_{\mathrm{g}} + \alpha_\ell\rho_\ell C_{p,\ell}\chi_\ell)c_{\mathrm{TF4}}^2, \tag{5.5}$$

$$\mathscr{P} = \frac{\alpha_{\mathrm{g}}\alpha_\ell\rho_{\mathrm{g}}\rho_\ell(v_{\mathrm{g}} - v_\ell)}{\alpha_\ell\rho_{\mathrm{g}} + \alpha_{\mathrm{g}}\rho_\ell}\frac{T}{L}\left(\rho_\ell C_{p,\ell}\chi_\ell\frac{\Psi_{\mathrm{g}}}{\rho_{\mathrm{g}}c_{\mathrm{g}}^2} - \rho_{\mathrm{g}}C_{p,\mathrm{g}}\chi_{\mathrm{g}}\frac{\Psi_\ell}{\rho_\ell c_\ell^2}\right)c_{\mathrm{TF4}}^2. \tag{5.6}$$

The eigenstructure for the general case is not accessible. However, when $v_{\mathrm{g}} = v_\ell$, we are able to find the exact eigenvalues of the system. For this, we write the characteristic polynomial of the matrix $A(U)$ where the velocities have been substituted with $v_{\mathrm{g}} = v_{\mathrm{m}}$ and $v_\ell = v_{\mathrm{m}}$

$$\Pi_{A, v_{\mathrm{g}} = v_\ell} = \mathrm{Det}(A(U_{v_{\mathrm{g}} = v_\ell}) - \lambda \cdot I_4), \tag{5.7}$$

where I_4 is the identity matrix of rank 4. This polynomial can be simplified to

$$\Pi_{A,v_{\mathrm{g}}=v_\ell} = (\lambda - v_{\mathrm{m}})^2 \cdot (\lambda - (v_{\mathrm{m}} + c_{\mathrm{TF4}})) \cdot (\lambda - (v_{\mathrm{m}} - c_{\mathrm{TF4}})), \tag{5.8}$$

which is solved by the eigenvalues presented in (5.1). □

5.2 Speed of sound in other models

The speed of sound of the five-equation model is given by (Martínez Ferrer *et al.* 2012). In order to express it in terms of the parameters used in the present article, we first derive a relation. In (Martínez Ferrer *et al.* 2012), the parameter

$$\zeta = \left(\frac{\partial T}{\partial p}\right)_s = -\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial s}\right)_p \tag{5.9}$$

is used. The triple product rule gives

$$\zeta = \frac{1}{\rho^2}\left(\frac{\partial p}{\partial s}\right)_\rho \Bigg/ \left(\frac{\partial p}{\partial \rho}\right)_s, \tag{5.10}$$

where

$$\left(\frac{\partial p}{\partial \rho}\right)_s = c^2, \tag{5.11}$$

and, from (Munkejord *et al.* 2009),

$$\left(\frac{\partial p}{\partial s}\right)_\rho = \Gamma \rho T. \tag{5.12}$$

Thus

$$\zeta = \frac{\Gamma T}{\rho c^2}. \tag{5.13}$$

The speed of sound in the five-equation model, taken from (Martínez Ferrer *et al.* 2012) and simplified, is

$$c_{\mathrm{TF5}} = \sqrt{\frac{\alpha_{\mathrm{g}}\rho_\ell + \alpha_\ell \rho_{\mathrm{g}}}{\rho_{\mathrm{g}}\rho_\ell \left(\frac{\alpha_{\mathrm{g}}}{\rho_{\mathrm{g}} c_{\mathrm{g}}^2} + \frac{\alpha_\ell}{\rho_\ell c_\ell^2} + \frac{\alpha_{\mathrm{g}}\rho_{\mathrm{g}} C_{p,\mathrm{g}} \alpha_\ell \rho_\ell C_{p,\ell} T \left(\frac{\Gamma_{\mathrm{g}}}{\rho_{\mathrm{g}} c_{\mathrm{g}}^2} - \frac{\Gamma_\ell}{\rho_\ell c_\ell^2} \right)^2}{\alpha_{\mathrm{g}}\rho_{\mathrm{g}} C_{p,\mathrm{g}} + \alpha_\ell \rho_\ell C_{p,\ell}} \right)}}. \tag{5.14}$$

We also know from (Flåtten & Lund 2011) the speed of sound in the drift-flux three-equation model. This model can be seen as the limit of the drift-flux four-equation model with instantaneous phase relaxation, or as the limit of the two-fluid four-equation model (3.35)–(3.38) with instantaneous velocity relaxation. This is obtained by summing equations (3.36) and (3.37) and assuming $v_{\mathrm{g}} = v_\ell$. After simplification, the speed of sound can be written

$$c_{\mathrm{DF3}} = \frac{1}{\sqrt{(\alpha_{\mathrm{g}}\rho_{\mathrm{g}} + \alpha_\ell\rho_\ell)\left(\frac{\alpha_{\mathrm{g}}}{\rho_{\mathrm{g}} c_{\mathrm{g}}^2} + \frac{\alpha_\ell}{\rho_\ell c_\ell^2} + T(\alpha_{\mathrm{g}}\rho_{\mathrm{g}} C_{p,\mathrm{g}} \chi_{\mathrm{g}}^2 + \alpha_\ell \rho_\ell C_{p,\ell} \chi_\ell^2)\right)}}. \tag{5.15}$$

5.3 Comparison of the speeds of sound

(Martínez Ferrer *et al.* 2012) compared the speeds of sound of four of the two-phase flow models in Figure 1 – the TF_6, TF_5, DF_5 and DF_4 models. They showed that the effect of the instantaneous relaxation of a given type on the mixture speed of sound is independent of the order in which relaxations are performed. For example, the effect of relaxing the velocity multiplies the speed of sound by a constant factor

$$\frac{c_{\mathrm{TF5}}}{c_{\mathrm{DF4}}} = \frac{c_{\mathrm{TF6}}}{c_{\mathrm{DF5}}} = \sqrt{(\alpha_{\mathrm{g}}\rho_{\mathrm{g}} + \alpha_\ell\rho_\ell)\left(\frac{\alpha_{\mathrm{g}}}{\rho_{\mathrm{g}}} + \frac{\alpha_\ell}{\rho_\ell}\right)}, \tag{5.16}$$

By rearranging the expression above, they also arrive at

$$\frac{c_{\mathrm{DF5}}}{c_{\mathrm{DF4}}} = \frac{c_{\mathrm{TF6}}}{c_{\mathrm{TF5}}}, \tag{5.17}$$

which shows that the same conclusion applies to the effect of thermal relaxation.

Now, in the present work, we derived TF_4 from the TF_5 model previously mentioned by performing instantaneous phase relaxation, and found its sound speed (5.2). By comparing it to the speed of sound in the DF_3 (5.15), we immediately see that we can extend the ratio relation (5.16) with

$$\frac{c_{\mathrm{TF4}}}{c_{\mathrm{DF3}}} = \frac{c_{\mathrm{TF5}}}{c_{\mathrm{DF4}}} = \frac{c_{\mathrm{TF6}}}{c_{\mathrm{DF5}}}, \tag{5.18}$$

which shows that the velocity relaxation once more has an independent effect on the speed of sound. From the above relation, we can deduce

$$\frac{c_{\mathrm{DF4}}}{c_{\mathrm{DF3}}} = \frac{c_{\mathrm{TF5}}}{c_{\mathrm{TF4}}}, \tag{5.19}$$

hence, the effect of phase relaxation on the sound speed is also independent from the order of the relaxation steps.

Using the results of (Martínez Ferrer *et al.* 2012) on the ordering of the speeds of sound, we can write from (5.18)

$$c_{\mathrm{DF3}} \leq c_{\mathrm{TF4}}. \tag{5.20}$$

Now, we take the difference between the two speeds of sound c_{TF4} and c_{TF5}, or more precisely the inverse of their squares, which gives

$$c_{\mathrm{TF4}}^{-2} - c_{\mathrm{TF5}}^{-2} = \frac{\rho_{\mathrm{g}}\rho_{\ell}}{\alpha_{\ell}\rho_{\mathrm{g}} + \alpha_{\mathrm{g}}\rho_{\ell}} \frac{T(\alpha_{\mathrm{g}}\rho_{\mathrm{g}}C_{p,\mathrm{g}}\chi_{\mathrm{g}} + \alpha_{\ell}\rho_{\ell}C_{p,\ell}\chi_{\ell})^2}{\alpha_{\mathrm{g}}\rho_{\mathrm{g}}C_{p,\mathrm{g}} + \alpha_{\ell}\rho_{\ell}C_{p,\ell}}. \tag{5.21}$$

This difference is always positive, which proves that

$$c_{\mathrm{TF4}} \leq c_{\mathrm{TF5}}. \tag{5.22}$$

Consequently, from (5.19)

$$c_{\mathrm{DF3}} \leq c_{\mathrm{DF4}}. \tag{5.23}$$

5.4 Subcharacteristic condition and model hierarchy

We can now extend the results SC1–SC4 from (Martínez Ferrer *et al.* 2012) by adding the two-fluid four-equation and the drift-flux three-equation models to the hierarchy. Following the argument of (Martínez Ferrer *et al.* 2012), as well as referring to (5.1) and to the eigenvalues of the drift-flux three-equation model in (Flåtten & Lund 2011), we can state the new results:

SC5: The model DF3 statisfies the subcharacteristic condition with respect to TF4.

SC6: The model DF3 statisfies the subcharacteristic condition with respect to DF4.

SC7: The model TF4 statisfies the weak subcharacteristic condition with respect to TF5.

Here we follow the definitions of the subcharacteristic and weak subcharacteristic conditions given by (Martínez Ferrer *et al.* 2012). For the two-fluid models, due to algebraic complexity, the general eigenvalues are not known. Therefore, we only discussed the case where the gas and liquid velocities are equal, which only proves a weak subcharacteristic condition.

6 Condition for hyperbolicity

The canonical model derived above, with $\Delta p = 0$, is generally not hyperbolic. Identically to the two-fluid six-equation model, the eigenvalues related to the volume-fraction waves are complex as soon as the gas and liquid velocities are different from each other ((Gidaspow 1974, Stuhmiller 1977)). The pressure difference term Δp has been added to make the model hyperbolic. In order to find an expression for Δp, we will use a perturbation method around the state where $v_{\mathrm{g}} = v_{\ell}$. Based on the experience from the two-fluid six-equation model ((Chang & Liou 2007, Evje & Flåtten 2003, Munkejord 2007, Munkejord *et al.* 2009, Paillère *et al.* 2003, Stuhmiller 1977)), we look for it in the form $\Delta p = \mathscr{C} \cdot (v_{\mathrm{g}} - v_{\ell})^2$. We know, from the section above, the speed of sound of the model, c_{TF4}. The variable defined as

$$\varepsilon = \frac{v_{\mathrm{g}} - v_{\ell}}{2 \cdot c_{\mathrm{TF4}}} \tag{6.1}$$

is small for subsonic velocities and is therefore suitable as a perturbation parameter. We first evaluate the characteristic polynomial

$$\Pi_A = \mathrm{Det}(A(U) - \lambda \cdot I_4), \tag{6.2}$$

where I_4 is the identity matrix of rank 4. In this polynomial, we make a variable change through

$$\lambda = \frac{v_{\mathrm{g}} + v_\ell}{2} + a \cdot c_{\mathrm{TF4}}, \tag{6.3}$$

where a is the new unknown. Then, all the occurences of the velocity are eliminated by substituting

$$v_{\mathrm{g}} = v_{\mathrm{m}} + \varepsilon \cdot c_{\mathrm{TF4}}, \tag{6.4}$$

$$v_\ell = v_{\mathrm{m}} - \varepsilon \cdot c_{\mathrm{TF4}}, \tag{6.5}$$

where v_{m} is the arithmetic average of v_{g} and v_ℓ. This is in compliance with the definition of ε (6.1).

Now, we perform a power-series expansion of the eigenvalues in terms of the degree of ε. To do so, the variable a is substituted by

$$a = \sum_{i=0}^{N} \left(b_i \cdot \varepsilon^i \right), \tag{6.6}$$

where N must be higher than the highest degree of ε that we wish in the expansion. Then we will sequentially solve

$$\mathrm{degree}(\Pi_A, \varepsilon, i) = 0 \tag{6.7}$$

for the coefficients b_i, starting from $i = 0$, where $\mathrm{degree}(\Pi_A, \varepsilon, i)$ returns the coefficient of the i^{th} degree of ε in $\Pi_A(\varepsilon)$.

The zeroth degree gives a fourth order equation in b_0,

$$\frac{\rho_{\mathrm{g}}^4 \rho_\ell^4 (\alpha_\ell \rho_{\mathrm{g}} + \alpha_{\mathrm{g}} \rho_\ell)^4 L^4}{(\rho_{\mathrm{g}} - \rho_\ell)^8 c_{\mathrm{TF4}}^4} (b_0 - 1)(b_0 + 1) b_0^2 = 0, \tag{6.8}$$

whose four solutions are $b_0 = -1$, $b_0 = 1$, and twice $b_0 = 0$. The first two give the approximate eigenvalues

$$\lambda = \frac{v_{\mathrm{g}} + v_\ell}{2} \pm c_{\mathrm{TF4}} + \mathscr{O}\left(\frac{v_{\mathrm{g}} - v_\ell}{2 \cdot c_{\mathrm{TF4}}} \right), \tag{6.9}$$

which are clearly the eigenvalues related to the pressure waves. The double solution $b_0 = 0$ corresponds to the volume-fraction waves, which are of interest here. For this wave family, we push to the next degree of the expansion. However, the first degree of the polynomial $\Pi_A(\varepsilon)$ vanishes when $b_0 = 0$. We then go to the second degree. Fortunately, b_2 vanishes from the second degree, and we are left with a second order equation in b_1

$$((\alpha_{\mathrm{g}} \rho_\ell + \alpha_\ell \rho_{\mathrm{g}}) b_1^2 + 2(\alpha_{\mathrm{g}} \rho_\ell - \alpha_\ell \rho_{\mathrm{g}}) b_1 + (\alpha_{\mathrm{g}} \rho_\ell + \alpha_\ell \rho_{\mathrm{g}}) - 4\mathscr{C}) \cdot \frac{\rho_{\mathrm{g}}^4 \rho_\ell^4 L^4 (\alpha_{\mathrm{g}} \rho_\ell + \alpha_\ell \rho_{\mathrm{g}})^3}{c_{\mathrm{TF4}}^4 (\rho_{\mathrm{g}} - \rho_\ell)^8} = 0. \tag{6.10}$$

The reduced discriminant of the equation is

$$\begin{aligned} \Delta &= (\alpha_{\mathrm{g}} \rho_\ell - \alpha_\ell \rho_{\mathrm{g}})^2 - (\alpha_{\mathrm{g}} \rho_\ell + \alpha_\ell \rho_{\mathrm{g}})(\alpha_{\mathrm{g}} \rho_\ell + \alpha_\ell \rho_{\mathrm{g}} - 4\mathscr{C}) \\ &= -4 \alpha_{\mathrm{g}} \alpha_\ell \rho_{\mathrm{g}} \rho_\ell + 4(\alpha_{\mathrm{g}} \rho_\ell + \alpha_\ell \rho_{\mathrm{g}}) \mathscr{C}. \end{aligned} \tag{6.11}$$

Therefore b_1 will only be real if

$$\mathscr{C} \geq \frac{\alpha_{\mathrm{g}} \alpha_\ell \rho_{\mathrm{g}} \rho_\ell}{\alpha_{\mathrm{g}} \rho_\ell + \alpha_\ell \rho_{\mathrm{g}}}, \tag{6.12}$$

which is the same constraint as the one obtained for the six-equation model ((Stuhmiller 1977)). The solutions are then

$$b_1 = \frac{-\alpha_{\mathrm{g}} \rho_\ell + \alpha_\ell \rho_{\mathrm{g}} \pm 2\sqrt{-\alpha_{\mathrm{g}} \alpha_\ell \rho_{\mathrm{g}} \rho_\ell + (\alpha_{\mathrm{g}} \rho_\ell + \alpha_\ell \rho_{\mathrm{g}}) \mathscr{C}}}{\alpha_{\mathrm{g}} \rho_\ell + \alpha_\ell \rho_{\mathrm{g}}}. \tag{6.13}$$

This gives the approximate eigenvalues for the volume-fraction waves

$$\lambda = \frac{v_{\mathrm{g}} + v_\ell}{2} + \frac{-\alpha_{\mathrm{g}} \rho_\ell + \alpha_\ell \rho_{\mathrm{g}} \pm 2\sqrt{-\alpha_{\mathrm{g}} \alpha_\ell \rho_{\mathrm{g}} \rho_\ell + (\alpha_{\mathrm{g}} \rho_\ell + \alpha_\ell \rho_{\mathrm{g}}) \mathscr{C}}}{\alpha_{\mathrm{g}} \rho_\ell + \alpha_\ell \rho_{\mathrm{g}}} \frac{v_{\mathrm{g}} - v_\ell}{2} + \mathscr{O}\left(\frac{v_{\mathrm{g}} - v_\ell}{2 \cdot c_{\mathrm{TF4}}} \right). \tag{6.14}$$

We deduce from the above that the model with the regularising term expressed as

$$\Delta p = \frac{\alpha_{\mathrm{g}} \alpha_\ell \rho_{\mathrm{g}} \rho_\ell}{\alpha_{\mathrm{g}} \rho_\ell + \alpha_\ell \rho_{\mathrm{g}}} (v_{\mathrm{g}} - v_\ell)^2 \tag{6.15}$$

is hyperbolic at first order around the state where $v_\mathrm{g} = v_\ell$. The same expression was previously derived for other models ((Chang & Liou 2007, Evje & Flåtten 2003, Munkejord 2007, Munkejord *et al.* 2009, Paillère *et al.* 2003, Stuhmiller 1977)). To make them actually hyperbolic when $v_\mathrm{g} \neq v_\ell$, the pressure difference in these models has commonly been defined as

$$\Delta p = \delta \frac{\alpha_\mathrm{g}\alpha_\ell\rho_\mathrm{g}\rho_\ell}{\alpha_\mathrm{g}\rho_\ell + \alpha_\ell\rho_\mathrm{g}}(v_\mathrm{g} - v_\ell)^2, \tag{6.16}$$

where $\delta > 1$ ((Chang & Liou 2007, Evje & Flåtten 2003, Munkejord *et al.* 2009, Paillère *et al.* 2003)).

7 Resonance

The two-fluid models are prone to resonance, which means that the eigenvector space collapses under some conditions, and the Jacobian of the fluxes becomes singular ((Isaacson & Temple 1990, Liu 1987b, Morin *et al.* 2012)). This is due to the eigenvectors related to the volume-fraction waves becoming parallel when the gas and liquid velocities are equal. The physical explanation is that the volume-fraction waves become identical – identical jump and propagation velocity. This is not a problem for numerical methods that do not use the eigenstructure of the system, because the two waves actually exist and are superimposed ((Morin *et al.* 2012)). However, this is problematic for numerical methods that use the eigenstructure, because it looks like information is lost. In this case, a fix can be used to overcome this issue, for example the one described by (Morin *et al.* 2012).

8 Conclusion

We have analysed a two-fluid four-equation model as the limit of a five-equation model when the phase relaxation becomes instantaneous. The phase relaxation source terms involve an interfacial momentum velocity, for which we found an expression respecting the second law of thermodynamics. This model was then put in quasilinear form by deriving the differentials of the primary variables. By this, we have extended previous works where these terms were treated as instantaneous relaxation source terms. Then the intrinsic speed of sound of the model has been extracted.

We have placed our model in a hierarchy of two-phase flow relaxation models. It has been proved in previous works that the subcharacteristic condition is satisfied for a part of this hierarchy. In the present work, we have proved that it is satisfied for the rest of our hierarchy.

Finally, we applied a perturbation method around the state where the gas and liquid velocities are equal. This helped deriving an expression for the pressure difference in the regularisation term which makes the model hyperbolic.

This model is ready to implement, using numerical methods for hyperbolic systems. One should nevertheless keep in mind that the model is prone to resonance, so that methods that use the eigenstructure of the system will require a fix when the gas and liquid velocities are equal or close to each other.

Acknowledgment

This publication has been produced with support from the BIGCCS Centre, performed under the Norwegian research program Centres for Envi-ronment-friendly Energy Research (FME). The authors acknowledge the following partners for their contributions: Aker Solutions, ConocoPhillips, Det Norske Veritas, Gassco, Hydro, Shell, Statoil, TOTAL, GDF SUEZ and the Research Council of Norway (193816/S60).

We are thankful to our colleagues Peder Kristian Aursand, Morten Hammer and Svend Tollak Munkejord for useful discussions.

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